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On the propagation of gaussian measures under the flow of Hamiltonian PDE's

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The Sobolev space $H^s(\mathbb{T}^d)$

• Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be a torus of dimension d.

• If $f : \mathbb{T}^d \to \mathbb{C}$ is a C^∞ function then for every $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

where

$$\widehat{f}(n) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx$$

then

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\widehat{f}(n)|^2 ,$$

where

$$\langle n\rangle^{2s} = (1 + n_1^2 + n_2^2 + \cdots + n_d^2)^s.$$

• One can define the Sobolev space $H^s(\mathbb{T}^d)$ as the closure of $C^\infty(\mathbb{T}^d)$ with respect to the H^s norm.

The Sobolev space $H^s(\mathbb{T}^d)$ (sequel)

• The norm H^s is induced from a natural scalar product

$$(f,g)_s = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \widehat{f}(n)\overline{\widehat{g}(n)}$$

which makes $H^s(\mathbb{T}^d)$ a Hilbert space.

• We have that

$$(f,g)_s = (2\pi)^{-d}\Big((1-\Delta)^s(f), g\Big),$$

where $(\cdot,\cdot)$ stands for the $L^2(\mathbb{T}^d)$ inner product and

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

is the Laplace operator.

The gaussian measure μ_s

- We wish to have a measure formally defined as

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as a measure on a suitable functional space.

- Formally

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du = Z^{-1} \exp\Big(-\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\widehat{u}(n)|^2\Big) \prod_{n\in\mathbb{Z}^d} d\,\widehat{u}(n)$$

and the last expression makes think about the well defined object

$$\prod_{n\in\mathbb{Z}} Z_n^{-1} \exp\big(-\langle n\rangle^{2s}|\widehat{u}(n)|^2\big) d\,\widehat{u}(n),$$

where we formally wrote

$$Z^{-1} = \prod_{n\in\mathbb{Z}} {Z_n}^{-1}\,.$$

The gaussian measure μ_s (sequel)

- Therefore, we may wish to define the measure μ_s, formally given by

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as the image measure by the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}^d} e^{in\cdot x}\, \frac{g_n(\omega)}{\langle n\rangle^s}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}^d}$ are i.i.d. complex gaussian random variables with mean 0 and variances 1, on a probability space $(\Omega, \mathcal{F}, p)$.

- **Question :** μ_s is a measure on which space ?

The gaussian measure μ_s (sequel)

- We can write for $N < M$

$$\Big\| \sum_{N \leq |n| \leq M} e^{in\cdot x} \, \frac{g_n(\omega)}{\langle n \rangle^s} \Big\|^2_{L^2(\Omega; H^\sigma(\mathbb{T}^d))} \simeq \sum_{N \leq |n| \leq M} \frac{\langle n \rangle^{2\sigma}}{\langle n \rangle^{2s}}$$

which tends to zero as $N \to \infty$, provided

$$\sigma < s - \frac{d}{2} \, .$$

- Therefore

$$\sum_{n \in \mathbb{Z}^d} e^{in\cdot x} \, \frac{g_n(\omega)}{\langle n \rangle^s} \in L^2(\Omega; H^\sigma(\mathbb{T}^d)) \, .$$

The gaussian measure μ_s (sequel)

• We conclude that the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}^d} e^{in\cdot x}\, \frac{g_n(\omega)}{\langle n\rangle^s}$$

defines a probability measure on $H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$. In addition, we will see later that

$$\mu_s(H^{s-\frac{d}{2}}(\mathbb{T}^d)) = 0\,.$$

• In particular

$$\mu_s(H^s(\mathbb{T}^d)) = 0\,.$$

• The triple $(H^\sigma(\mathbb{T}^d), H^s(\mathbb{T}^d), \mu_s)$ defines an abstract Wiener space.

• In this constriction $H^s(\mathbb{T}^d)$ is canonical (called the Cameron-Martin space) but $H^\sigma(\mathbb{T}^d)$ is not, it may be replaced for instance by $W^{\sigma,\infty}(\mathbb{T}^d)$.

A first property of μ_s

• Recall that μ_s is a probability measure on $H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$.

• Let $l \in (H^\sigma(\mathbb{T}^d))^\star$, i.e. $l\colon H^\sigma(\mathbb{T}^d) \to \mathbb{C}$, linear and continuous.

• Let $\mu_s \circ l^{-1}$ be the transport on $\mathbb{C}$ by l of the measure μ_s. It results directly from the definition that $\mu_s \circ l^{-1}$ is a centered complex gaussian with a variance depending on l.

• More precisely if l is identified with $v \in H^{-\sigma}(\mathbb{T}^d)$ (i.e. $l(x) = (x, v)$) then $\mu_s \circ l^{-1}$ is the law of the random variable (with values in $\mathbb{C}$)

$$\sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)\overline{\widehat{v}(n)}}{\langle n \rangle^s}$$

which is a centered complex gaussian with variance

$$\sum_{n \in \mathbb{Z}^d} \frac{|\widehat{v}(n)|^2}{\langle n \rangle^{2s}} = \sum_{n \in \mathbb{Z}^d} \frac{|\widehat{v}(n)|^2 \langle n \rangle^{-2\sigma}}{\langle n \rangle^{2s-2\sigma}} < \infty$$

because $\sigma < s - \frac{d}{2}$ and $v \in H^{-\sigma}(\mathbb{T}^d)$.

Conclusion

• Therefore μ_s is a gaussian measure on $H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$ according to the text books definition.

The covariance operator associated with μ_s

• We define a bilinear map T on $(H^\sigma(\mathbb{T}^d))^\star \times (H^\sigma(\mathbb{T}^d))^\star$ by

$$T(l_1, l_2) = \mathbb{E}_{\mu_s}(l_1(x)\overline{l_2(x)}) = \int_{H^\sigma(\mathbb{T}^d)} l_1(x)\overline{l_2(x)}\mu_s(dx)\,.$$

If we identify l_1, l_2 with $v_1, v_2 \in H^{-\sigma}(\mathbb{T}^d)$ respectively then

$$T(l_1, l_2) = \mathbb{E}\Big(\sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)\overline{\widehat{v_1}(n)}}{\langle n\rangle^s}, \sum_{n\in\mathbb{Z}^d} \frac{\overline{g_n(\omega)}\widehat{v_2}(n)}{\langle n\rangle^s}\Big)$$
$$= (2\pi)^{-d}\overline{((1-\Delta)^{-s}(v_1), v_2)}.$$

• In this sense, we may say that $(2\pi)^{-d}(1-\Delta)^{-s}$ is the "covariance matrix" of μ_s.

• Recall that formally

$$\mu_s = Z^{-1}\, e^{-\|u\|^2_{H^s}}\, du$$

and

$$\|u\|^2_{H^s} = (2\pi)^{-d}\Big((1-\Delta)^s(u), u\Big).$$

A finer almost property with respect to μ_s

- Let $\langle D_x \rangle^\sigma$ be a Fourier multiplier defined by

$$\langle D_x \rangle^\sigma f(x) = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^\sigma \widehat{f}(n)\, e^{in\cdot x},$$

corresponding to a fractional derivation of order σ. Set

$$\varphi(\omega, x) = \sum_{n \in \mathbb{Z}^d} e^{in\cdot x}\, \frac{g_n(\omega)}{\langle n \rangle^s}$$

which describes the support of μ_s. Then we have

Proposition 1

Let $\sigma < s - \frac{d}{2}$. Then

$$\langle D_x \rangle^\sigma (\varphi(\omega, x)) \in C(\mathbb{T}^d)$$

almost surely. In other words $\langle D_x \rangle^\sigma (u) \in C(\mathbb{T}^d)$, μ_s almost surely.

- **Remark.** A priori, we only know that $\langle D_x \rangle^\sigma (\varphi(\omega, x)) \in L^2(\mathbb{T}^d)$.

Proof of the proposition

• For every x, we have that $g_n(\omega)\, e^{in\cdot x}$ is again a standard complex gaussian (invariance under rotations of gaussian vectors).

• Next, using the independence of g_n we get that for a fixed $x \in \mathbb{T}^d$,

$$\langle D_x \rangle^\sigma (\varphi(\omega, x))$$

is a standard complex gaussian with variance

$$\sum_{n \in \mathbb{Z}^d} \frac{\langle n \rangle^{2\sigma}}{\langle n \rangle^{2s}} < \infty.$$

• Consequently $\forall\, p < \infty$, $\| \langle D_x \rangle^\sigma (\varphi(\omega, x)) \|_{L^p(\Omega)}$ is finite and independent of x. Consequently

$$\langle D_x \rangle^\sigma (\varphi(\omega, x)) \in L^p(\Omega \times \mathbb{T}^d)$$

Thanks Fubini $\langle D_x \rangle^\sigma (\varphi(\omega, x)) \in L^p(\mathbb{T}^d)$ almost surely for all $p < \infty$.

• We finally use the Sobolev embedding (the restriction on σ is open).

Optimality of the restriction $\sigma < s - \frac{d}{2}$.

Proposition 2
$u \notin H^{s-\frac{d}{2}}(\mathbb{T}^d)$, *$\mu_s$ almost surely.*

Proof. Set again

$$\varphi(\omega, x) = \sum_{n \in \mathbb{Z}^d} e^{in \cdot x} \frac{g_n(\omega)}{\langle n \rangle^s}$$

We need to study the probability of the event A defined by

$$A = \{\omega : \|\varphi(\omega, \cdot)\|_{H^{s-\frac{d}{2}}} < \infty\}.$$

The event A belongs to the asymptotic σ-algebra obtained from the independent σ-algebras generated from g_n because the property $\|u\|_{H^{s-\frac{d}{2}}} < \infty$ depends only on $(1 - \Pi_N)u$ for every $N \in \mathbb{N}$, where Π_N is the Dirichlet projector.

Sequel of the proof

• Therefore by the Kolmogorov zero-one law, we have that

$$p(A) \in \{0, 1\}.$$

• We suppose that the last probability is 1 and we look for a contradiction. Set $\sigma = s - \frac{d}{2}$. If $p(A) = 1$ then by the dominated convergence

$$\lim_{N\to\infty} \int_{\Omega} e^{-\|\pi_N \varphi(\omega,\cdot)\|^2_{H^\sigma}} dp(\omega) = \int_{\Omega} e^{-\|\varphi(\omega,\cdot)\|^2_{H^\sigma}} dp(\omega) > 0\,. \qquad (1)$$

• We will show that

$$\lim_{N\to\infty} \int_{\Omega} e^{-\|\pi_N \varphi(\omega,\cdot)\|^2_{H^\sigma}} dp(\omega) = 0$$

which will be in a contradiction with (1).

• Using the independence, we can write

$$\int_{\Omega} e^{-\|\pi_N \varphi(\omega,\cdot)\|^2_{H^\sigma}} dp(\omega) = \prod_{|n|\leq N} \int_{\mathbb{R}^2} e^{-\langle n\rangle^{-2s}(x^2+y^2)\langle n\rangle^{2\sigma}} e^{-(x^2+y^2)} \frac{dxdy}{\pi}\,.$$

Recall that $2\sigma - 2s = -d$.

Sequel of the proof

- Now, if we set

$$\theta := \int_{x^2+y^2\le 1} e^{-(x^2+y^2)} \frac{dxdy}{\pi} \in (0,1)$$

and we have

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)\langle n\rangle^{-d}} e^{-(x^2+y^2)} \frac{dxdy}{\pi}$$

$$\le \theta + \int_{x^2+y^2>1} e^{-\langle n\rangle^{-d}} e^{-(x^2+y^2)} \frac{dxdy}{\pi}$$

$$\le \theta + e^{-\langle n\rangle^{-d}}(1-\theta) = 1-(1-\theta)(1-e^{-\langle n\rangle^{-d}}) \,.$$

- Now, we observe that

$$\lim_{N\to\infty} \sum_{|n|\le N} (1-e^{-\langle n\rangle^{-d}}) = \infty$$

because

$$\lim_{N\to\infty} \sum_{|n|\le N} \langle n\rangle^{-d} = \infty.$$

This completes the proof.

Almost sure products with respect to μ_s

• When solving nonlinear PDE we need to give sense of products of low regularity functions and also distributions.

• When $s > \frac{d}{2}$ we can readily define μ_s almost surely the operation

$$(u_1, u_2) \longmapsto u_1(x) \times u_2(x), \quad x \in \mathbb{T}^d$$

because $\mu_s(C(\mathbb{T}^d)) = 1$, thanks to the previous proposition.

• The situation is radically different for $s \leq \frac{d}{2}$ because in this case the support of μ_s is not of classical functions and we deal with a random distributions (or random fields).

Almost sure products with respect to μ_s for $s < \frac{d}{2}$

Let $s < \frac{d}{2}$. The random distribution

$$\varphi(\omega, x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n(\omega)}{\langle n \rangle^s} e^{in\cdot x}, \qquad \frac{d}{4} < s < \frac{d}{2}$$

belongs only to a Sobolev space of negative regularity and therefore it is hard to define an object like $|\varphi(\omega, x)|^2$. For example, thanks to Parseval, the zero Fourier coefficient of $|\varphi(\omega, x)|^2$ should be

$$\sum_{n \in \mathbb{Z}^d} \frac{|g_n(\omega)|^2}{\langle n \rangle^{2s}}$$

which is a.s. divergent. However, it turns out that the zero Fourier coefficient is the only obstruction and it is possible, *after a renormalisation*, to define $|\varphi(\omega, x)|^2$ and even to compute its Sobolev regularity.

Almost sure products with respect to μ_s for $s < \frac{d}{2}$ (sequel)

• Fix $\sigma < s - \frac{d}{2}$ (close to $s - \frac{d}{2}$). Consider the partial sums

$$\varphi_N(\omega, x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^{\alpha}} \, e^{in\cdot x} \in C^{\infty}(\mathbb{T}^d)$$

and write

$$|\varphi_N(\omega, x)|^2 = \sum_{|n| \leq N} \frac{|g_n(\omega)|^2}{\langle n \rangle^{2s}} + \sum_{\substack{n_1 \neq n_2 \\ |n_1|, |n_2| \leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^{s} \langle n_2 \rangle^{s}} \, e^{i(n_1 - n_2)\cdot x}.$$

• The first term (the zero Fourier coefficient) contains all the singularity while the second has an a.s. limit in $H^{2\sigma}(\mathbb{T}^d)$.

Almost sure products with respect to μ_s for $s < \frac{d}{2}$ (sequel)

- Consequently, we set

$$c_N := \mathbb{E}\Big(\sum_{|n|\leq N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2s}}\Big) = \mathbb{E}(|\varphi_N(\omega,x)|^2) = \sum_{|n|\leq N} \frac{1}{\langle n\rangle^{2s}} \sim N^{d-2s},$$

and we define the renormalised partial sums

$$|\varphi_N(\omega,x)|^2 - c_N = \sum_{|n|\leq N} \frac{|g_n(\omega)|^2-1}{\langle n\rangle^{2s}} + \sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^s\langle n_2\rangle^s}\, e^{i(n_1-n_2)\cdot x}.$$

- Thanks to the independence of g_n we have

$$\mathbb{E}\Big(\Big|\sum_{|n|\leq N} \frac{|g_n(\omega)|^2-1}{\langle n\rangle^{2s}}\Big|^2\Big) = \sum_{|n|\leq N} \frac{4}{\langle n\rangle^{4s}},$$

which has a limit as $N\to\infty$ when $s > d/4$.

Almost sure products with respect to μ_s for $s < \frac{d}{2}$ (sequel)

- Another use of the independence yields that

$$\mathbb{E}\Big(\Big\| \sum_{\substack{n_1 \neq n_2 \\ |n_1|,|n_2| \leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^s \langle n_2 \rangle^s}\, e^{i(n_1-n_2)\cdot x} \Big\|_{H^{2\sigma}}^2 \Big)$$

is equal to

$$\Big\| \sum_{\substack{n_1 \neq n_2 \\ |n_1|,|n_2| \leq N}} \langle n_1 - n_2 \rangle^{\sigma} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^s \langle n_2 \rangle^s}\, e^{i(n_1-n_2)\cdot x} \Big\|_{L^2(\Omega \times T^d)}$$

which is bounded by

$$C \sum_{n_1,n_2} \frac{\langle n_1 - n_2 \rangle^{4\sigma}}{\langle n_1 \rangle^{2s} \langle n_2 \rangle^{2s}}\,.$$

The last sum converges as far as $-4\sigma + 4s > 2d$, which is equivalent to our assumption $\sigma < s - \frac{d}{2}$. Hence we proved that :

Proposition 3

The sequence $\left(|\varphi_N(\omega,x)|^2 - c_N\right)_{N\geq 1}$ *has a limit in* $L^2(\Omega; H^{2\sigma}(\mathbb{T}))$*.*

This limit is by definition the renormalisation of $|\varphi(\omega,x)|^2$*, i.e. after a renormalisation we can give a sense of* $|u|^2$*,* μ_s *almost surely.*

Remarks

• Using more involved arguments, we can also show the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T}^d)$ of the sequence

$$\Big(|\varphi_N(\omega,x)|^2 - c_N\Big)_{N\geq 1}.$$

• Since $\sigma<0$ the norm in $H^{2\sigma}(\mathbb{T}^d)$ is weaker than in $H^{\sigma}(\mathbb{T}^d)$ (where $\varphi_N(\omega,x)$ is defined).

• Informally : the square of the modulus of an element of H^σ is in $H^{2\sigma}$, after a renormalisation.

• This is a remarkable probabilistic phenomenon, in the heart of the study of evolution partial differential equations in the presence of randomness in Sobolev spaces of negative indexes.

• If $s=1$ then the restriction

$$s>\frac{d}{4}$$

is OK for $d=2,3$ but not for $d\geq 4$. This is related to the existence of the Φ^4_d theories for $d=1,2,3$ and the triviality for $d\geq 4$.

Remarks (sequel)

• We can replace the gaussians with much more general random variables.

• We can also replace the sequence

$$\frac{1}{\langle n\rangle^s}$$

with a more general sequence (c_n), i.e. we can consider

$$\sum_{n\in\mathbb{Z}^d} c_n\, g_n(\omega)\, e^{in\cdot x}$$

instead of

$$\sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s}\, e^{in\cdot x}$$

but I am not aware of the optimal regularity of the renormalised square in function of the sequence (c_n).

Control on the μ_s a.s. divergence of the H^σ, $\sigma < s - \frac{d}{2}$ norm

• • Recall that we have shown that

$$\|u\|_{H^{s-\frac{d}{2}}} = \infty, \quad \mu_s \text{ almost surely.}$$

However, we have that

$$\|\Pi_N \varphi(\omega,\cdot)\|_{H^\sigma}^2 = \sum_{|n|\leq N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2s-2\sigma}}$$

and therefore for $\sigma > s - \frac{d}{2}$,

$$\mathbb{E}\Big(\|\Pi_N \varphi(\omega,\cdot)\|_{H^\sigma}^2\Big) = \sum_{|n|\leq N} \frac{1}{\langle n\rangle^{2s-2\sigma}} \sim N^{d-2s+2\sigma}$$

• As in the previous analysis, we can show that

$$\|\Pi_N \varphi(\omega,\cdot)\|_{H^\sigma}^2 - \mathbb{E}\Big(\|\Pi_N \varphi(\omega,\cdot)\|_{H^\sigma}^2\Big)$$

has a limit as $N \to \infty$, provided $4s - 4\sigma > d$, i.e. $\sigma < s - \frac{d}{4}$.

Conclusion

• Therefore for σ between $s - \frac{d}{2}$ and $s - \frac{d}{4}$ we control the divergence of the H^σ norm μ_s almost surely.

• This fact may play an important role in the analysis of PDE's with data distributed according to μ_s (quasi-invariant measures for the nonlinear wave equation and invariant measures for the Benjamin-Ono equation).

The Cameron-Martin theorem

• **Question :** How behaves μ_s under transformations ?

Theorem 4 (Cameron-Martin 1944)
Let $f \in H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$ and let μ_f be the image of μ_s under the map from $H^\sigma(\mathbb{T}^d)$ to $H^\sigma(\mathbb{T}^d)$ defined by

$$u \longmapsto f + u \, .$$

Then μ_f is absolutely continuous with respect to μ_s if and only if

$$f \in H^s(\mathbb{T}^d).$$

• Recalling that formally

$$d\mu_s(u) = Z^{-1} \, e^{-\|u\|_{H^s}^2} \, du$$

we may expect that

$$\frac{d\mu_f(u)}{d\mu_s(u)} = e^{-\|f\|_{H^s}^2 - 2(u,f)_s} \, ,$$

where $(\cdot,\cdot)_s$ stands for the H^s scalar product.

Proof of the Cameron-Martin theorem for μ_s

• Let $f \in H^s(\mathbb{T}^d)$. Since we expect that the Radon-Nykodim derivative is $\exp\Big(-\|f\|_{H^s}^2 - 2(u,f)_s\Big)$ the first issue is to show that $(u,f)_s < \infty$, μ_s almost surely which is equivalent to

$$\sum_{n\in\mathbb{Z}^d} \langle n\rangle^s\, g_n(\omega)\, \overline{\widehat{f}(n)} < \infty, \qquad \text{a.s.}$$

which directly results directly from the independence and $f \in H^s(\mathbb{T}^d)$.

• We however need also to show that $\exp\Big(-2(u,f)_s\Big)$ is μ_s integrable.

• For that purpose we write for $\lambda > 0$,

$$\mu_s(u\colon |(u,f)_s| > \lambda) = p\Big(\omega\colon \Big|\Big(\sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s}\, e^{in\cdot x}\,,\, f\Big)_s\Big| > \lambda\Big)$$

$$= p\Big(\omega\colon \Big|\sum_{n\in\mathbb{Z}^d} \langle n\rangle^s\, g_n(\omega)\, \overline{\widehat{f}(n)}\Big| > \lambda\Big) \leq C e^{-C\lambda^2}$$

because

$$\sum_{n\in\mathbb{Z}^d} \langle n\rangle^s\, g_n(\omega)\, \overline{\widehat{f}(n)} \in \mathcal{N}_{\mathbb{C}}(0,\sigma^2),\ \sigma^2 = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\widehat{f}(n)|^2 < \infty\,.$$

Proof of the Cameron-Martin theorem for μ_s (sequel)

• Let now $f \notin H^s(\mathbb{T}^d)$. Then there is $g \in H^s$ such that $(f,g)_s = \infty$. Consider the set

$$A = \{u \in H^\sigma \,:\, (g,u)_s < \infty\}.$$

We already checked that $\mu_s(A) = 1$ (replace f by g in the discussion of the previous slide). The image of A under our shift is the set B defined by

$$B = \{u + f, \quad u \in A\}.$$

Clearly $A \cap B = \emptyset$ and therefore $\mu_s(B) = 0$. Thus we found a set of measure 1 which is sent by the shift by f map to a set of measure 0. This completes the proof.

The Ramer theorem

• For $\sigma < s - \frac{d}{2}$, let us consider a diffeomorphism Φ on $H^\sigma(\mathbb{T}^d)$ of the form

$$\Phi(u) = u + F(u),$$

where $F\colon H^\sigma(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$. Suppose that for $u \in H^\sigma(\mathbb{T}^d)$,

$$DF[u]\colon H^s(\mathbb{T}^d) \to H^s(\mathbb{T}^d)$$

is Hilbert-Schmidt.

Theorem 5 (Ramer 1974)

Under the above assumption μ_s is quasi-invariant under Φ.

The Ramer theorem applies to

$$F(u) = \varepsilon(1-\Delta)^{-d/2-\delta}(u^2), \quad \delta > 0,\ |\varepsilon| \ll 1,$$

i.e. d-smoothing is needed.

• Unfortunately such a strong smoothing is not (directly) available if Φ is the flow of a nonlinear PDE.

• The Ramer theorem is optimal in the class of general diffeomorphisms of the above form.

How behaves μ_s under the flow of a Hamiltonian PDE ?

The free Schrödinger evolution

• The linear Schrödinger equation reads

$$(i\partial_t + \Delta)u = 0, \quad u|_{t=0} = u_0\,. \tag{2}$$

If u_0 is given by

$$u_0(x) = \sum_{n\in\mathbb{Z}^d} c_n\, e^{in\cdot x}$$

then the solution of (2) is given by

$$u(t,x) = \sum_{n\in\mathbb{Z}^d} c_n\, e^{-it|n|^2}\, e^{in\cdot x}.$$

• We write

$$u(t,x) = e^{it\Delta}(u_0),$$

i.e. the map $e^{it\Delta}$ generates the solutions of (2).

Invariance of μ_s under the free Schrödinger evolution

Proposition 6
Let $S(t) = e^{it\Delta}$. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T}^d)$ to $H^\sigma(\mathbb{T}^d)$ defined by $u \longmapsto S(t)(u)$. Then $\mu_s(t) = \mu_s$.

Proof. We have that

$$S(t)\Big(\sum_{n\in\mathbb{Z}^d} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}\Big) = \sum_{n\in\mathbb{Z}^d} e^{inx}\,\frac{e^{-itn^2}g_n(\omega)}{\langle n\rangle^s}$$

which has the same distribution as

$$\sum_{n\in\mathbb{Z}^d} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}$$

because $e^{-itn^2}g_n(\omega)$ has the same distribution as $g_n(\omega)$ (invariance of complex gaussians by rotations). This completes the proof.

A remark

• Even in $1d$, for a fixed sequence $(c_n)_{n\in\mathbb{Z}}$ the free Schrödinger evolution

$$\sum_{n\in\mathbb{Z}} c_n\, e^{inx}\, e^{-itn^2}$$

may have a complicated behaviour depending on the nature of the number t (leading to interesting number theory considerations) but the statistical behaviour under μ_s is the same for each time t.

Transport of μ_s under nonlinear transformations

Question : How behaves μ_s under the flow of the nonlinear Schrödinger equation (NLS) ? Let us start by the dispersionless model :

Theorem 7 (Oh-Sosoe-Tz. (2017))

Let $d = 1$, $s \geq 1$ be an integer and $0 < \sigma < s - 1/2$. Let $\rho_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves

$$i\partial_t u = |u|^2 u, \quad u|_{t=0} = u_0 \,. \tag{3}$$

Then for $t \neq 0$, the measure $\rho_s(t)$ is not absolutely continuous with respect to μ_s.

- The solution of (3) is given by

$$u(t,x) = u_0(x)\, e^{-it|u_0(x)|^2} \tag{4}$$

and the idea behind the proof is to show that a typical regularity property of the data resulting from the iterated logarithm law associated with μ_s is destroyed by the time oscillation in formula (4).

Transport of μ_s under nonlinear transformations (sequel)

But we also have :

Theorem 8 (Deng-Sun-Tz. 2022)

Let $s > 2$ and $1 \leq \sigma < s - 1$. Let $p \geq 2$ be an even integer. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T}^2)$ to $H^\sigma(\mathbb{T}^2)$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves the $2d$ nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u = |u|^p u, \quad u|_{t=0} = u_0 \,. \tag{5}$$

Then $\mu_s(t)$ is absolutely continuous with respect to μ_s. In other words, μ_s is quasi-invariant under the flow of (5). In particular for fixed t, x the law of $u(t,x)$ has a density with respect to the Lebesgue measure on $\mathbb{C}$.

Remark. We know that (5) is globally well-posed in $H^\sigma(\mathbb{T}^2)$, $\sigma \geq 1$, thanks to the work by Bourgain (1992).

Remarks

• Previously, we had similar results for NLS in $1d$, for the nonlinear wave equations in dimensions ≤ 3 (with energy sub-critical nonlinearities), for the gKdV equation and for BBM type models.

• The first result for measures in negative Sobolev spaces is by Oh-Seong in the context of 4NLS.

• The $3d$ NLS does not seem out of reach ...

• Depending on the equation, we have more or less informations on the resulting Radon-Nykodim derivatives. The first result identifying the Radon-Nykodim derivative as a suitable $L^p(\mu_s)$ function is by Debussche-Tsutsumi.

• For $s = 1$, the quasi-invariance may be a consequence from the invariance of the Gibbs measure. However, in many case the renormalizations change considerably the model and the result makes no connection with the smooth solutions of the considered equation.

A corollary (L^1 stability for the corresponding Liouville equation)

Theorem 9

Let $s > 2$. *Let* $f_1, f_2 \in L^1(d\mu_s)$ *and let* $\Phi(t)$ *be the flow of*

$$(i\partial_t + \Delta)u = |u|^{2p}u, \quad u|_{t=0} = u_0\,,$$

defined μ_s *a.s. Then for every* $t \in \mathbb{R}$, *the transports of the measures*

$$f_1(u)d\mu_s(u), \quad f_2(u)d\mu_s(u)$$

by $\Phi(t)$ *are given by*

$$F_1(t,u)d\mu_s(u), \quad F_2(t,u)d\mu_s(u)$$

respectively, for suitable $F_1(t,\cdot), F_2(t,\cdot) \in L^1(d\mu_s)$. *Moreover*

$$\|F_1(t) - F_2(t)\|_{L^1(d\mu_s)} = \|f_1 - f_2\|_{L^1(d\mu_s)}\,.$$

• Local in time bounds for other distances are obtained in a recent work by work by Forlano-Seong. There are many further things to be understood.

Methods

• Roughly speaking, presently, we have two different methods to prove this kind of quasi-invariance results :

• **Method 1 :** Using the *time oscillations* (dispersive estimates).

• **Method 2 :** Using the *random oscillations* (in the spirit of the analysis we did in the beginning of the lectures).

• In both methods, we do not study directly the evolution of the gaussian measure μ_s but the evolution of ρ_s defined by

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where $R_s(u)$ is a suitable correction and where χ is a continuous function with a compact support and where $H(u)$ is the Hamiltonian of the equation under consideration (conserved by the flow). We formally have

$$e^{-R_s(u)} d\mu_s(u) = Z^{-1} e^{-R_s(u)} e^{-\|u\|_{H^s}^2} du = Z^{-1} e^{-E_s(u)} du\,,$$

where

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)\,.$$

Methods (sequel)

• The correction $R_s(u)$ in the energy functional

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)$$

is of fundamental importance and there are different intuitions behind its construction : normal form reductions, traces of complete integrability, modulated energies, ...

• Interestingly, in some cases the construction of $R_s(u)$ requires renormalisation arguments.

• However, an important feature is that we *do not renormalise the equation which stays always the same*. Instead, we consider renormalised functionals associated with the equation with data distributed according to a gaussian field.

On method 1

- Let $\Phi(t)$ be the flow of the PDE under consideration.
- Formally the transported measure is given by

$$Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, du = Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, e^{E_s(u)} e^{-E_s(u)} du$$

which can be interpreted as the (relatively) well defined object

$$e^{-\left(E_s(\Phi(t)(u)) - E_s(u)\right)}\, \chi(H(u)) e^{-R_s(u)} d\mu_s(u)\,.$$

- Therefore we hope that the Radon-Nykodim derivative of the transport of ρ_s is given by

$$e^{-\left(E_s(\Phi(t)(u)) - E_s(u)\right)}$$

- **Problem :** In $E_s(\Phi(t)(u)) - E_s(u)$ both terms are strongly diverging on the support of μ_s but the hope is to find some cancellations thanks to PDE smoothing estimates.

On method 1 (sequel)

• More precisely, one can write

$$E_s(\Phi(t)(u)) - E_s(u) = \int_0^t \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau} d\tau.$$

Set

$$G_s(\tau) = \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau}.$$

We will be done, if we can prove that

$$\Big| \int_0^t G_s(\tau) d\tau \Big| \leq C_{H(u)} \|u\|^{\theta}_{H^{s-\frac{d}{2}-}},$$

for a suitable choice of $R_s(u)$ and for a suitable number θ.

• If E_s is a conserved quantity (Gibbs measures) then $G_s = 0$ and one expects an invariant measure. However, this may not be true at the level of the approximated finite dimensional models and a serious difficulty may appear (cf. works by Nahmod-Oh-Rey Bellet-Staffilani, Tz.-Visciglia, Genovese-Luca-Valeri, ...).

On method 1 (sequel)

• If $\theta < 2$ the Randon-Nykodim density is indeed given by

$$e^{-\left(E_s(\Phi(t)(u)) - E_s(u)\right)}$$

in the sense that it is the natural limit of the corresponding (perfectly well defined) finite dimensional densities.

• If $\theta \geq 2$, we can define the Radon-Nykodim density of the transport of

$$\exp\left(- \|u\|^m_{H^{s-\frac{d}{2}-}} \right) \chi(H(u))\, e^{-R_s(u)} d\mu_s(u),$$

where $m \gg 1$ (depending on θ).

• **Remark.** It would be interesting to replace

$$\left| \int_0^t G_s(\tau) d\tau \right| \leq C_{H(u)} \|u\|^\theta_{H^{s-\frac{d}{2}-}},$$

with more subtle estimates.

On method 2

- Let $A \subset H^\sigma(\mathbb{T})$ be a measurable set.
- Recall that

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where χ is a continuous function with a compact support and $H(u)$ is the Hamiltonian of the equation under consideration.

- Then

$$\frac{d}{dt}\rho_s(\Phi(t)(A))\Big|_{t=\bar{t}} = \frac{d}{dt}\rho_s(\Phi(t)(\Phi(\bar{t})(A)))\Big|_{t=0}$$

which is **formally** equal to

$$\int_{\Phi(\bar{t})(A)} \frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0} d\rho_s(u)$$
$$\leq \Big\|\frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0}\Big\|_{L^p(\rho_s)} \Big(\rho_s(\Phi(\bar{t})(A))\Big)^{1-\frac{1}{p}}$$

On method 2 (sequel)

• We would be done if we show that

$$\left\| \frac{d}{dt} E_s(\Phi(t)(A)) \Big|_{t=0} \right\|_{L^p(\rho_s)} \leq Cp, \quad p \gg 1 \,. \tag{6}$$

In the proof of the last inequality we only exploit the random oscillations of the initial data.

• Important observation : if we are only interested in the qualitative statement of quasi-invariance then in (6) we can suppose that A included in a bounded set of a Banach space $\mathcal{H}$ which is of full measure such that the PDE under consideration is globally well posed in $\mathcal{H}$ (existence, uniqueness and persistence of regularity).

• Let us **formally** show how we use (6) (similarly to the uniqueness for 2d Euler) to get the quasi-invariance. Set

$$x(t) = \rho_s(\Phi(t)(A))\,.$$

Thanks to (6) we have

$$\dot{x}(t) \leq Cp(x(t))^{1-\frac{1}{p}}$$

On method 2 (sequel)

Therefore

$$\frac{d}{dt}\Big((x(t))^{\frac{1}{p}}\Big) \le C\,.$$

• An integration yields

$$(x(t))^{\frac{1}{p}} - (x(0))^{\frac{1}{p}} \le Ct$$

Therefore, if $x(0) = 0$ then

$$x(t) \le (Ct)^p$$

which goes to zero as $p \to \infty$, provided $Ct < 1$.

• Since the constant C is uniform we can iterate the last argument and achieve any time.

• The above argument may become rigorous if we use some approximation arguments resulting from the Cauchy problem theory of the equation under consideration.

Final remarks

• Basically, it may look that Method 2 performs better for equations with weaker dispersion.

• I do not see yet an efficient way to combine Method 1 and Method 2 ...

• In the work on 2d NLS with Deng and Sun, we follow Method 2 with several key novelties. One of them is that thanks to the structure of the resonant set we can use a normal from reduction and then use the time oscillations via the Strichartz estimates for the linear equation (a similar idea was used in my work with Hani-Pausader-Visciglia on solutions of NLS with growing higher Sobolev norms).

• We are not able so far to use the recent refined resolution ansatz (as the random averaging operators) in the context of quasi-invariance of gaussian measures. It would be very interesting to clarify whether it may be possible. This is what I am presently trying to understand ...

2D NLS analysis, the setup

- Write

$$v(t) = e^{-it\Delta}u(t), \quad v(t) = \sum_k v_k(t)e^{ik\cdot x}.$$

- If $u(t)$ solves $i\partial_t u + \Delta u = |u|^2 u$, then

$$\partial_t v_k = \frac{1}{i} \sum_{k_1 - k_2 + k_3 = k} e^{-it\Phi(\vec{k})} v_{k_1} \overline{v}_{k_2} v_{k_3},$$

where

$$\Phi(\vec{k}) := |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2 = 2(k_1 - k_2)\cdot(k_2 - k_3).$$

- We have

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 = -\frac{1}{4}\mathrm{Im} \sum_{\substack{k_1 - k_2 + k_3 - k_4 = 0 \\ k_2 \neq k_1, k_3}} \psi_{2s}(\vec{k}) e^{-it\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2} v_{k_3} \overline{v}_{k_4},$$

$$\psi_{2s}(\vec{k}) = |k_1|^{2s} - |k_2|^{2s} + |k_3|^{2s} - |k_4|^{2s}.$$

• Set

$$\mathcal{N}_{0,t}(v) = \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \psi_{2s}(\vec{k}) \frac{e^{-it\Phi(\vec{k})}}{-i\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4},$$

$$\mathcal{R}_{0,t}(v) = \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})= 0}} \psi_{2s}(\vec{k}) v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4}$$

$$\mathcal{R}_{1,1,t}(v) = 2\sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it\Phi(\vec{k})} \sum_{p_1-p_2+p_3=k_1} e^{-it\Phi(\vec{p})} v_{p_1}\overline{v}_{p_2}v_{p_3}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4},$$

$$\mathcal{R}_{1,2,t}(v) = -2\sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it\Phi(\vec{k})} \sum_{q_1-q_2+q_3=k_2} e^{it\Phi(\vec{q})} v_{k_1}\overline{v}_{q_1}v_{q_2}\overline{v}_{q_3}v_{k_3}\overline{v}_{k_4}.$$

• Defining

$$E_{s,t}(v) := \frac{1}{2}\|v\|^2_{H^s} + \frac{1}{4}\mathrm{Im}\mathcal{N}_{0,t}(v)$$

we obtain that along the NLS flow, we have

$$\frac{d}{dt}E_{s,t}(v) := \frac{1}{4}\mathrm{Im}\Big[\mathcal{R}_{1,1,t}(v) + \mathcal{R}_{1,2,t}(v) - \mathcal{R}_{0,t}(v)\Big]$$

• Let us look at the simplest (resonant) term

$$\mathcal{R}_{0,t}(v) := \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4}.$$

• W.L.O.G., we assume that $v_{k_j} = \widehat{P_{N_j}v}(k_j)$ and $N_{(1)} \geq N_{(2)} \geq N_{(3)} \geq N_{(4)}$ are the rearrangement of N_1, N_2, N_3, N_4.

• We have $|\psi_{2s}(\vec{k})| \lesssim N_{(1)}^{2s-2} N_{(3)}^2$ and therefore

$$|\mathcal{R}_{0,t}(v)| \lesssim N_{(1)}^{2s-2} N_{(3)}^2 \int_0^{2\pi}\int_{\mathbb{T}^2} e^{it\Delta} f_1 \cdot \overline{e^{it\Delta} f_2} e^{it\Delta} f_3 \cdot \overline{e^{it\Delta} f_4} dt dx,$$

where $\widehat{f_j}(k_j) = |v_{k_j}|$. The space-time integral can be treated using the bilinear Strichartz estimate. Due to the unavoidable loss $N_{(3)}^{0+}$, we have

$$|\mathcal{R}_{0,t}(v)| \lesssim \|\mathbf{P}_{N_{(1)}} v\|_{H^{s-1}} \|\mathbf{P}_{N_{(2)}} v\|_{H^{s-1}} \|\mathbf{P}_{N_{(3)}} v\|_{H^{2+}} \|\mathbf{P}_{N_{(4)}} v\|_{L^2}.$$

• No matter how large s is, the above estimate is not enough for our need, as $v \in H^{(s-1)-}$ almost surely. Nevertheless, we are ϵ-close to what we expect (for s large).

Exploiting the random oscillation

• By Method II, what we are allowed reduce the estimate to $t=0$ and average on the support of the measure. So we have access to the probability toolbox: Wiener chaos estimate: $l-$linear Gaussian sum:

$$\mathcal{T}_l := \sum_{k_1,\cdots,k_l} c_{k_1,\cdots,k_l} g_{k_1}(\omega)\cdots g_{k_l}(\omega),$$

for any $p\geq 2$, $\|\mathcal{T}_l\|_{L^p_\omega}\leq Cp^{\frac{l}{2}}\|\mathcal{T}_l\|_{L^2_\omega}$.

• The pairing contributions $(k_1=k_2,k_3=k_4),(k_1=k_4,k_2=k_3)$ in $\mathcal{R}_{0,t}(v)$ disappear by taking the imaginary part, it is reduced to estimate

$$p^2\Big\|\sum_{\substack{k_1-k_2+k_3-k_4=0,\\ k_2\neq k_1,k_3\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k})\mathbf{1}_{|k_j|\sim N_j}\frac{g_{k_1}(\omega)\overline{g}_{k_2}(\omega)g_{k_3}(\omega)\overline{g}_{k_4}(\omega)}{\langle k_1\rangle^s\langle k_2\rangle^s\langle k_3\rangle^s\langle k_4\rangle^s}\Big\|_{L^2_\omega}$$

Consider the worst case, say $N_1\sim N_2\gg N_3+N_4=O(1)$, the above quantity can be crudely bounded by $p^2N_{(1)}^{2s-2}\cdot N_{(1)}^{-2s+1}=p^2N_{(1)}^{-1}$. By interpolating with the deterministic bound in the last slide, we conclude that $\|\mathrm{Im}\mathcal{R}_{0,t}(v)|_{t=0}\|_{L^p_\omega}\leq Cp$.

The key cancellation

- The treatment for $\mathcal{N}_{0,t}(v)$ follows from the similar analysis + resonance decomposition according to the value of $\Phi(\vec{k})$.

- However, the estimate for the second generations $\mathcal{R}_{1,j,t}(v), j = 1, 2$ requires another algebraic cancellation.

- The reason is that in the high-high-low-low-low-low regime, the most problematic contribution is the paring of two dominant frequencies living in different generations. These types of pairing prevent us to gain from the Wiener chaos.

The key cancellation (sequel)

- Written in formula, these two pairing configurations are:

$$\mathcal{S}_{1,1,1}(v) :=$$

$$4 \sum_{k_1 \neq k_2} |v_{k_2}|^2 \sum_{\substack{|k_3|+|k_4 \ll |k_1|,|k_2| \\ |p_2|+|p_3| \ll |k_1|,|k_2| \\ k_3-k_4=k_2-k_1 \\ p_2-p_3=k_2-k_1}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it(|k_3|^2-|k_4|^2+|p_2|^2-|p_3|^2)} v_{k_3} \overline{v}_{k_4} \overline{v}_{p_2} v_{p_3},$$

and

$$\mathcal{S}_{1,1,2}(v) :=$$

$$4 \sum_{k_1,k_3} |v_{k_3}|^2 \sum_{\substack{|k_2|+|k_4| \ll |k_1|,|k_3| \\ |p_1|+|p_3| \ll |k_1|,|k_3| \\ p_1+p_3=k_1+k_3 \\ k_2+k_4=k_1+k_3}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{it(|k_2|^2+|k_4|^2-|p_1|^2-|p_3|^2)} \overline{v}_{k_2} \overline{v}_{k_4} v_{p_1} v_{p_3}.$$

The key cancellation (sequel)

• To understand the hidden cancellation, for $\mathcal{S}_{1,1,1}(v)$, one can think about the sum is taken over $|k_3|, |k_4|, |p_2|, |p_3| = O(1)$, then

$$\frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} \approx \frac{|k_1|^{2s} - |k_2|^{2s}}{|k_1|^2 - |k_2|^2},$$

then the second sum in the definition of $\mathcal{S}_{1,1,1}$ is completely decoupled as $|\cdots|^2$ and we deduce that $\mathcal{S}_{1,1,1}$ is almost real.

Thank you for your attention !