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# Universality results for a class of nonlinear wave equations

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## The microscopic model

- For $N \geq 1$, consider as a microscopic model the wave dynamics

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta} \Pi_N V'(u) = 0, \ (t,x) \in \mathbb{R} \times \mathbb{T}_N^2 \, , \alpha \leq 1,$$

where $\theta > 0$ , $|D_x|^{2\alpha} = (-\Delta)^{\alpha}$ and

$$\mathbb{T}_N^2 \equiv (\mathbb{R}/2\pi N)^2, \ \ V(u) = \sum_{j=0}^{m} a_j u^{2j}, \quad m \geq 2, \, a_m > 0.$$

- $\Pi_N$ is a Dirichlet projector defined as

$$\Pi_N \Big( \sum_{k \in \mathbb{Z}^2} \widehat{f}(k/N) \ e^{i\frac{k \cdot x}{N}} \Big) = \sum_{|k| \leq N} \widehat{f}(k/N) \ e^{i\frac{k \cdot x}{N}}$$

for

$$f(x) = \sum_{k \in \mathbb{Z}^2} \widehat{f}(k/N) \ e^{i\frac{k \cdot x}{N}}$$

a function on $\mathbb{T}_N^2 = (\mathbb{R}/2\pi N \mathbb{Z})^2$.

## The microscopic model (sequel)

• Without the term $N^{-\theta}\Pi_N V'(u)$ in

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta}\Pi_N V'(u) = 0,$$

we have $N$ linear waves (each Fourier coefficient) oscillating independently.

• Indeed, the solution of

$$\partial_t^2 u + |D_x|^{2\alpha} u = 0, \quad u(0,x) = u_0, \ \partial_t u(0,x) = u_1$$

is given by

$$u(t,x) = \cos(t|D_x|^{\alpha})\, u_0 + \frac{\sin(t|D_x|^{\alpha})}{|D_x|^{\alpha}}\, u_1\,.$$

• We have

$$\cos(t|D_x|^{\alpha})(e^{i\frac{k\cdot x}{N}}) = \cos(t|k/N|^{\alpha})\, e^{i\frac{k\cdot x}{N}}$$

and a similar formula for the sin contribution.

• **The question we study :** Understand how the weak nonlinear interaction $N^{-\theta}\Pi_N V'(u)$ modifies the free evolution for $N \gg 1$.

## On the nature of the nonlinear interaction

• The presence of $\Pi_N$ in

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta} \Pi_N V'(u) = 0,\ (t,x) \in \mathbb{R} \times \mathbb{T}_N^2\,,$$

is essential for the existence of the dynamics.

• Indeed, consider

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta} u^{2k+1} = 0,\ (t,x) \in \mathbb{R} \times \mathbb{T}_N^2\,. \tag{1}$$

Then for $k > \frac{\alpha}{1-\alpha}$ (1) is an energy supercritical problem and it is not clear at all that there is a well-defined flow, even for smooth data.

• More precisely, the energy controls the $H^\alpha$ norm while the scaling invariant norm is $H^{1-\frac{\alpha}{k}}$. Then for $\alpha < 1$,

$$1 - \frac{\alpha}{k} > \alpha \quad \Longleftrightarrow \quad k > \frac{\alpha}{1-\alpha}\,.$$

## The initial data

• We therefore consider

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{-\theta} \Pi_N V'(u) = 0,\ (t,x) \in \mathbb{R} \times \mathbb{T}_N^2$$

with gaussian initial data

$$u(0,x) = \phi_N(x)\ , \quad (\partial_t u)(0,x) = \psi_N(x),$$

where

$$\phi_N(x) = \frac{1}{(2\pi)^2} N^{\alpha-1} \sum_{|k|\le N} \frac{g_k(\omega)}{\langle k\rangle^\alpha}\ e^{i\frac{k\cdot x}{N}}\ ,$$

with $\langle k\rangle^\alpha := (1+|k|^{2\alpha})^{\frac{1}{2}}$ and

$$\psi_N(x) = \frac{1}{(2\pi)^2} N^{-1} \sum_{|k|\le N} h_k(\omega)\ e^{i\frac{k\cdot x}{N}}\ .$$

• Here $g_k$ and $h_k$ are standard complex Gaussians such that $g_k = \overline{g_{-k}}$, $h_k = \overline{h_{-k}}$ and otherwise independent.

• The initial position $\phi_N(x)$ and the initial speed $\psi_N(x)$ are gaussians with variances $\sim 1$, independent of $x$.

## On the structure of the initial data

• The initial data is, very roughly speaking, essentially of the form

$$\frac{1}{(2\pi)^2}\,N^{-1}\sum_{|k|\leq N} f(k/N)\,g_k(\omega)\;e^{i\frac{k\cdot x}{N}}\,.$$

for a suitable function $f:\mathbb{R}^2\rightarrow\mathbb{R}$.

• We have

$$f(x)=\frac{1}{\langle x\rangle^{\alpha}},\qquad f(x)=1$$

for the initial position and the initial velocity respectively.

• This is a very restrictive choice related to the support of the corresponding Gibbs measures.

## Assumptions on the potential

- Note that $\phi_N$ has a stationary Gaussian distribution. More precisely

$$\phi_N(x) \sim \mathcal{N}(0, \sigma_N^2) \ , \qquad \forall\, x \in \mathbb{T}_N^2,$$

where for $\alpha < 1$

$$\sigma_N^2 = \frac{1}{4\pi^2 N^{2(1-\alpha)}} \sum_{|k| \le N} \frac{1}{\langle k \rangle^{2\alpha}} = \underbrace{\frac{1}{4\pi^2} \int_{|\xi|<1} \frac{1}{|\xi|^{2\alpha}} \mathrm{d}\xi}_{\sigma^2} + \mathcal{O}(N^{-2(1-\alpha)}) \ .$$

- Let $\mu = \mathcal{N}(0, \sigma^2)$, and

$$\langle V \rangle(z) := \int_{\mathbb{R}} V(z+y) \mu(\mathrm{d}y)$$

be the average of $V$ under $\mu$. Our main assumption on the polynomial $V$ is the criticality and the positivity of its averaged version $\langle V \rangle$.

## Assumptions on the potential (sequel)

More precisely, we suppose that $V$ is an even polynomial, given by

$$V(z) = \sum_{j=0}^{2m} a_j z^{2j}\,, \quad m \geq 2$$

and we assume that the averaged polynomial

$$\langle V \rangle(z) := \int_{\mathbb{R}} V(z+y)\mu(\mathrm{d}y)$$

satisfies

1. $\langle V \rangle''(0) = 0$.

2. $\langle V \rangle(z) - \langle V \rangle(0) > 0$ for all $z \neq 0$.

## Assumptions on the potential (sequel)

- We have that

$$\langle V\rangle(z) = \sum_{j=0}^{m} \overline{a}_j z^{2j}$$

where

$$\overline{a}_j = \frac{1}{(2j)!}\mathbb{E}\Big[V^{(2j)}\Big(\mathcal{N}(0,\sigma^2)\Big)\Big] \, ,$$

and we can compute

$$\overline{a}_j = \frac{1}{(2j)!}\sum_{k=j}^{m} \frac{(2k)!}{(2k-2j)!!} \cdot a_k \cdot \sigma^{2(k-j)} \, .$$

- Then the the first assumption is $\overline{a}_1 = 0$ and the second one is

$$\sum_{j=2}^{m} \overline{a}_j \, z^{2(j-2)} > 0, \quad \forall\, z \in \mathbb{R}.$$

$\boxed{\text{Are there potentials satisfying the assumptions ?}}$

• If we fix $a_2 > 0$, ... ,$a_m > 0$, we can find $a_1 < 0$ such that our assumptions on $V$ are satisfied. For example

$$V(z) = z^6 - 45\sigma^2 z^2$$

satisfies the assumptions.

• We can find $V \geq 0$ such that our assumptions are satisfied.

## The macroscopic model

• Define the rescaled process $u_N$ on $\mathbb{R} \times \mathbb{T}^2$ by

$$u_N(t,x) := N^{1-\alpha} u(N^\alpha t, Nx) \ .$$

• The spatial domain of $u_N$ becomes the standard torus $\mathbb{T}^2$ and the equation for $u_N$ then becomes

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + N^{1+\alpha-\theta} \Pi_N V'(N^{\alpha-1} u_N) = 0$$

with initial datum

$$u_N(0,x) = N^{1-\alpha} \phi_N(Nx) = \frac{1}{(2\pi)^2} \sum_{|k|\le N} \frac{g_k(\omega)}{\langle k \rangle^\alpha} \ e^{ik\cdot x}$$

and

$$\partial_t u_N(0,x) = N \psi_N(Nx) = \frac{1}{(2\pi)^2} \sum_{|k|\le N} h_k(\omega) \ e^{ik\cdot x}.$$

## The macroscopic model

• In order for the cubic power in the macroscopic dynamics

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + N^{1+\alpha-\theta} \Pi_N V'(N^{\alpha-1} u_N) = 0$$

to have $\mathcal{O}(1)$ coefficient, one necessarily needs to set $\alpha$ and $\theta$ such that

$$1 + \alpha - \theta = 3(1-\alpha) \quad \Longleftrightarrow \quad \theta = 4\alpha - 2 \,.$$

• Therefore we expect that under such a scaling at macroscopic level the dynamics is governed by a "cubic equation" (even of there is no cubic term in the polynomial $V'$ !).

• The criticality condition on the averaged potential assures that the linear term has a limit.

• This condition also guarantees that one does not need to do further renormalizations at macroscopic level, as it assures that all divergent terms cancel out themselves. Roughly speaking, if one wants to see interesting (nontrivial) behavior at macroscopic level then the averaged potential has to satisfy the criticality condition.

## Solving the cubic equation

- Consider

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + \Pi_N (u_N)^3 = 0,$$

posed on $\mathbb{T}^2$ with gaussian initial data

$$(u_N(0,x), \partial_t u_N(0,x)) = \frac{1}{(2\pi)^2} \sum_{|k|\leq N} \Big( \frac{g_k(\omega)}{\langle k \rangle^\alpha}\ e^{ik\cdot x}, h_k(\omega)\ e^{ik\cdot x} \Big). \qquad (2)$$

**Theorem 1**
*Let $1 > \alpha > \frac{8}{9}$. Then there is a divergent sequence $(c_N)_{N\geq 1}$ such that the solutions of*

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + \Pi_N \Big( (u_N)^3 - c_N u_N \Big) = 0$$

*with initial data* (2) *converge almost surely in the sense of distribution on $\mathbb{R} \times \mathbb{T}^2$, as $N \to \infty$.*

## The full model

- We now consider the full macroscopic problem

$$\partial_t^2 u + |D_x|^{2\alpha} u + N^{3(1-\alpha)} \Pi_N V'(N^{\alpha-1} u) = 0.$$

- We have that

$$N^{3(1-\alpha)} \Pi_N V'(N^{\alpha-1} u) = \Pi_N \Big( N^{4(1-\alpha)} V(N^{\alpha-1} u) \Big)' = \Pi_N (V_N'(u)),$$

where

$$V_N(u) := N^{4(1-\alpha)} V(N^{\alpha-1} u).$$

Therefore, we have

$$V_N'(u) = \sum_{j=1}^{m} (2j) \overline{a}_{j,N} N^{-(2j-4)(1-\alpha)} H_{2j-1}(u; \tilde{\sigma}_N^2),$$

where $H_l(x;\sigma)$ denotes the Hermite polynomial of degree $l$ and

$$\tilde{\sigma}_N^2 := \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}^2, |k| \le N} \frac{1}{\langle k \rangle^{2\alpha}} \, .$$

## $\boxed{\text{The full model (sequel)}}$

- Recall that the Hermite polynomials are defined by

$$e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma).$$

In particular

$$H_1(x;\sigma) = x,\ H_2(x,\sigma) = x^2 - \sigma,\ H_3(x,\sigma) = x^3 - 3\sigma x.$$

## The full model

• We have that the coefficients $\overline{a}_{j,N}$ appearing in

$$V_N'(u) = \sum_{j=1}^{m} (2j)\overline{a}_{j,N} N^{-(2j-4)(1-\alpha)} H_{2j-1}(u_N; \tilde{\sigma}_N^2)$$

satisfy

$$\lim_{N\to+\infty} \overline{a}_{j,N} = \overline{a}_j \,, \quad \forall\, j \in \mathbb{N}.$$

• We have that $\overline{a}_1 = 0$ and that limit of $N^{2(1-\alpha)}\overline{a}_{1,N}$ as $N \to +\infty$ exists. More precisely :

**Proposition 2**

*Assume that $\alpha \in \left(\frac{1}{2}, 1\right)$. There exists an absolute constant $\lambda_0 \in \mathbb{R}$, such that as $N \to \infty$,*

$$\overline{a}_{1,N} = \overline{a}_1 + \lambda_0 N^{-2(1-\alpha)} + O(N^{-1}) + O(N^{-4(1-\alpha)}).$$

## Main result

**Theorem 3**

*Suppose that* $1 > \alpha > \frac{8}{9}$. *Let* $s < \alpha - 1$ *and suppose that* $V$ *satisfies our assumptions. Let* $u_N$ *be the solution of*

$$\partial_t^2 u_N + |D_x|^{2\alpha} u_N + \Pi_N V_N'(u_N) = 0,$$

*with initial data*

$$(u_N(0,x), \partial_t u_N(0,x)) = \frac{1}{(2\pi)^2} \sum_{|k| \le N} \Big( \frac{g_k(\omega)}{\langle k \rangle^\alpha} \, e^{ik\cdot x}, h_k(\omega) \, e^{ik\cdot x} \Big). \qquad (3)$$

*There is* $\lambda > 0$ *and a divergent sequence* $(c_N)_{N\ge 1}$ *such that the solutions of*

$$\partial_t^2 v_N + |D_x|^{2\alpha} v_N + \Pi_N(\lambda (v_N)^3 - c_N v_N) = 0$$

*with initial data* (3) *converge almost surely in the sense of distribution on* $\mathbb{R} \times \mathbb{T}^2$, *as* $N \to \infty$ *and satisfy*

$$\lim_{N\to\infty} \|u_N - v_N\|_{\mathcal{C}([-T,T],H^s(\mathbb{T}^2))} = 0, \; \forall\, T > 0.$$

## Comments

• We have that $\lambda = 4\overline{a}_2$ (which is necessarily $> 0$).

• We can have more precise convergence of $u_N - v_N$ by decomposing $v_N$ in a random low regularity term plus a smoother contribution. The smoother contribution converges in positive Sobolev regularity norms.

• For $V$ of high degree (depending on $\alpha$), the data is of supercritical regularity, even with respect to the threshold of probabilistic well-posedness proposed by Deng-Nahmod-Yue.

• This type of weak universality was first studied by Hairer-Quastel for deriving KPZ equation from microscopic growth models. Then it was extended to other parabolic singular SPDE by many authors.

## Comments (sequel)

• Our techniques can be used to extend the weak universality result of Gubunelli-Koch-Oh for the 2D stochastic nonlinear wave equation to the stochastic nonlinear fractional wave equation with space-time white noise, formally written as

$$\partial_t^2 u + |D_x|^{2\alpha} u + \partial_t u + \lambda u^{\diamond 3} = \xi, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^2$$

when $\alpha > \frac{8}{9}$. Gubunelli-Koch-Oh treat the case $\alpha = 1$.

• The weak universality result of Gubunelli-Koch-Oh is a consequence of the almost sure global well-posedness for the two-dimensional nonlinear wave equation ($\alpha = 1$) with *any order nonlinearity*, while for the fractional wave equation with $\alpha < 1$, the situation is radically different.

• In a recent joint work with Liu-Wang, we proved similar universality results in the $3d$ case. A challenging open problem remains open.

## The Gibbs measure

- Let $\mu$ be the gaussian measure induced by the map

$$\omega \longmapsto \frac{1}{(2\pi)^2} \sum_{k\in\mathbb{Z}^2} \frac{g_k(\omega)}{\langle k\rangle^\alpha} e^{ik\cdot x} .$$

- Let $\nu_N$ be the probability measure given by

$$\nu_N(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_N} e^{-\int_{\mathbb{T}^2} \left(V_N(\Pi_N\phi) - 1/2((\Pi_N\phi)^2 - \tilde{\sigma}_N^2)\right)\mathrm{d}x} \mu(\mathrm{d}\phi) .$$

The measure $\nu_N$ is well defined as long as $a_m > 0$.

- If $\lambda := \overline{a}_2 > 0$, then for any $c \in \mathbb{R}$ the measure

$$\nu(c)(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}} e^{-\lambda \int_{\mathbb{T}^2} \phi^{\diamond 4}\mathrm{d}x + c\int_{\mathbb{T}^2} \phi^{\diamond 2}\mathrm{d}x} \mu(\mathrm{d}\phi)$$

is also well-defined, where $\phi^{\diamond k}$ denotes the $k$-th Wick power of $\phi$ with respect to the Gaussian structure induced by $\mu$.

## The Gibbs measure (sequel)

**Theorem 4**

*Let $\alpha \in (\frac{3}{4}, 1)$. Suppose that $V$ satisfies our assumptions. Then*

$$\sup_N |\log \mathcal{Z}_N| < +\infty$$

*and there exists $c \in \mathbb{R}$ such that $\nu_N$ converges to $\nu(c)$ in total variation. In particular, $\nu_N(A)$ converges to $\nu(c)(A)$ for every Borel set $A$.*

- The restriction $\alpha > \frac{3}{4}$ is natural in the sense that in this range, one can define the $\phi^4$ measure by an absolutely continuous density with respect to the Gaussian measure $\mu$. The fourth Wick power $\phi^{\diamond 4}$ fails to exist under $\mu$ when $\alpha = \frac{3}{4}$, in which case one expects to end up with a measure (after further renormalizations) that is mutually singular with respect to $\mu$.

- **Remark.** We expect "triviality" for $\alpha = \frac{1}{2}$ ...

## On the optimality of our assumptions

**Proposition 5**

*If there exists* $\theta \in \mathbb{R}$ *such that*

$$\sum_{j=1}^{m} \overline{a}_j \theta^{2(j-2)} < 0$$

*then there exists* $c > 0$ *such that*

$$\log \mathcal{Z}_N > c N^{4(1-\alpha)}, \quad \forall\, N \in \mathbb{N}.$$

$\boxed{\text{Mains steps in the convergence proof}}$

The convergence proof contains two ingredients :

1. A priori bounds resulting from the invariance of the Gibbs measures associated both with the cubic equation and with the full model.

2. Dispersive effects giving $L_t^2 L_x^\infty$ local bounds.

## A first use an invariant measure

• We have that for any $\delta > 0$

$$\Big\| \sum_{|k|\leq N} \frac{g_k(\omega)}{\langle k\rangle^{\alpha}}\, e^{ik\cdot x} \Big\|_{L^\infty_x} \leq C_\delta N^{1-\alpha+\delta}$$

in a set of residual probability $\lesssim$ exp$(-N^{\theta})$ for some $\theta > 0$.

• As in the work by Bourgain-Bulut thanks to invariant measure considerations, we can propagate this information to the full solution $u_N$.

• This is unfortunately not sufficient to pass into the limit in terms like

$$u_N^3 \left(N^{-(1-\alpha)} u_N\right)^{2k+1}$$

for $k \gg 1$ because of small losses of power of $N$ in $N^{-(1-\alpha)} u_N$.

## Exploiting the dispersive effect

• We can overcome the above difficulty by writing

$$u_N^3\left(N^{-(1-\alpha)}u_N\right)^{2k+1} = N^{-(1-\alpha)}u_N^4\left(N^{-(1-\alpha)}u_N\right)^{2k}$$

and by exploiting the $L_t^2 L_x^\infty$ control coming from Strichartz estimates.

• This leads to local in time convergence.

• The global in time convergence crucially relies on the a priori bounds on the global cubic dynamics. These bounds are again relying on invariant measure considerations but this time for the limit dynamics.

• This essentially explains the basic idea behind the proof.

## A final remark

• As in the work by Bourgain-Bulut or by Burq-Tz. in the local convergence, we have inequalities of type

$$\dot{x}_N(t) \leq C_\delta(\log(N))^\delta x_N(t),$$

i.e. we allow a slow growth of order $\exp\Big((\log(N))^\delta\Big)$, $\delta < 1$.

• This is compensated by the convergence of $x_N(0)$ which is of order $N^{-\theta}$ for some $\theta > 0$.

## $\boxed{\text{Perspectives}}$

- Similar results for other dispersive models.
- The Benjamin-Ono equation seems a challenging case.
- Triviality results when the assumptions on $V$ are not satisfied.
- The critical case.
- More general initial data.
- Universal models coming from interactions of higher degree.

Thank you very much !