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Probabilistic well-posedness and Gibbs measure evolution for the non linear Schrödinger equation on the two dimensional sphere

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A classical result

- Let (M, g) be a compact $2d$ Riemannian manifold, $\partial M = \emptyset$.
- Non linear Schrödinger equation or NLS on M writes as follows :

$$(i\partial_t + \Delta)u = |u|^2 u.$$

- Conservation law

$$E(u) = \|u\|^2_{H^1(M)} + \frac{1}{2}\int_M |u|^4,$$

where

$$\|u\|_{H^s(M)} = \|(1-\Delta)^{s/2}(u)\|_{L^2(M)}.$$

The energy $E(u)$ almost controls the $L^\infty(M)$ norm and therefore :

Theorem 1 (Brézis-Gallouêt 1980)
NLS is globally well-posed in $H^s(M)$, $s \geq 1$.

- Using smoothing coming from the dispersion, we can extend this result to $3d$ (Burq-Gérard-Tz. 2001).
- **Question :** Can we extend this result to $H^s(M)$, $s < 1$?

Why we ask this question ?

• The formal object

$$Z^{-1}\, e^{-H(u)}\, du = Z^{-1}\, e^{-\frac{1}{2}\int |u|^4}\, e^{-\|u\|^2_{H^1}}\, du \tag{1}$$

is the Gibbs measure associated with NLS.

• Rigorously, after a renormalization (1) is defined as a measure absolutely continuous with respect to the measure defined by the map

$$\omega \longmapsto \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\langle \lambda_n \rangle}\, \varphi_n(x),$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots,$

$$-\Delta \varphi_n = \lambda_n^2 \varphi_n,$$

$(\varphi_n)_{n\geq 0}$ being an orthonormal bases of $L^2(M)$ and $(g_n)_{n\geq 0}$ being a family of i.i.d. standard complex gaussians.

• **Remark:** By definition $\langle \lambda_n \rangle = (1+\lambda_n^2)^{\frac{1}{2}}$.

Why we ask this question ? (sequel)

- We have that

$$\psi_\alpha(x,\omega) := \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\langle\lambda_n\rangle^\alpha}\,\varphi_n(x) \in H^s(M),\ s<\alpha-1,\ \text{almost surely.} \quad (2)$$

- Moreover $\psi_\alpha \notin H^{\alpha-1}(M)$ almost surely.

Proof of (2). By definition, we can write

$$\int_\Omega |(1-\Delta)^{s/2}\,\psi_\alpha(x,\omega)|^2 dp(\omega) = \sum_{n=0}^{\infty} \frac{|\varphi_n(x)|^2}{\langle\lambda_n\rangle^{2\alpha-2s}}$$

which can be estimated by

$$\sum_{N-\text{dyadic}} N^{2s-2\alpha} \sum_{\lambda_n\sim N} |\varphi_n(x)|^2 \lesssim \sum_{N-\text{dyadic}} N^{2s-2\alpha+2} < \infty,$$

thanks to the Hörmander spectral function theorem

$$\sum_{\lambda_n\sim N} |\varphi_n(x)|^2 \lesssim N^2$$

(N^2 becomes N^d in dim d). This ends the proof of (2).

• Therefore, if one wants to solve NLS with initial data typical with respect to the Gibbs measure then one needs to show well-posedness for a large dense set of initial data in $H^s(M) \cap (L^2(M))^c$, for all $s < 0$ (the previous analysis with $\alpha = 1$).

• The main issue is the local well-posdness because thanks to work by Bourgain we know that relatively quantified probabilistic local well-posedness with respect to the Gibbs measure implies the global well-posedness by exploiting measure propagation arguments instead of conservation laws.

Review of the deterministic well-posedness

Theorem 2 (Burq-Gérard-Tz. 2001)
NLS is locally well-posed in $H^s(M)$, $s > 1/2$. For every datum of size $R \geq 1$ in $H^s(M)$ the solution is defined on $[0, T]$ with $T \approx R^{-\alpha}$, $\alpha > 0$.

• If we only use energy estimates we would have the previous result for $s > 1$.

• The main ingredient in the proof is the estimate

$$\| \exp(it\Delta)(f) \|_{L^2([0,1];L^\infty(M))} \leq C \|f\|_{H^s(M)}, \quad s > 1/2. \tag{3}$$

• Estimate (3) fails for $s < 1/2$ in the case of S^2.

• If one wishes to improve the above result then one should not try to control the quantity $\|u\|_{L^2([0,T];L^\infty(M))}$ which is the analogue of $\|\nabla u\|_{L^1([0,T];L^\infty(M))}$ appearing in some fluid dynamics PDE.

Review of the deterministic well-posedness

Theorem 3 (Bourgain 1992)
NLS is locally well-posed in $H^s(\mathbb{T}^2)$, $s > 0$, where $\mathbb{T}^2$ denotes the flat torus. For every datum of size $R \geq 1$ in $H^s(\mathbb{T}^2)$ the solution is defined on $[0, T]$ with $T \approx R^{-\alpha}$, $\alpha > 0$.

• The main ingredient in the proof is the estimate

$$\|P_Q \exp(it\Delta)(f)\|_{L^4([0,1]\times\mathbb{T}^2)} \leq C_\varepsilon \, |Q|^\varepsilon \, \|f\|_{L^2(\mathbb{T}^2)}, \quad \varepsilon > 0, \tag{4}$$

where $Q \subset \mathbb{R}^2$ is a square and P_Q is the projector

$$P_Q(f) = \sum_{n\in\mathbb{Z}^2\cap Q} \widehat{f}(n)\, e^{in\cdot x} \,.$$

• In a remarkable recent work Herr-Kwak extended the solution obtained in Theorem 3 globally in time.

Theorem 4 (Burq-Gérard-Tz. 2003)
NLS is locally well-posed in $H^s(S^2)$, $s > 1/4$. For every datum of size $R \geq 1$ in $H^s(S^2)$ the solution is defined on $[0, T]$ with $T \approx R^{-\alpha}$, $\alpha > 0$. Moreover, for $s < 1/4$ NLS on S^2 fails to be semi-linearly well-posed.

• The main ingredients in the proof are arithmetic properties of the sequence $(n^2)_{n\in\mathbb{Z}}$ and the estimate

$$\|P_n P_m\|_{L^2(S^2)} \leq C(1 + \min(n, m))^{1/4} \|P_n\|_{L^2(S^2)} \|P_m\|_{L^2(S^2)} \qquad (5)$$

where P_n and P_m are spherical harmonics of degrees n and m respectively and C is an absolute constant (we do not know the optimal one even for $m = n$).

• For $s < 1/4$, we use that estimate (5) is asymptotically optimal by testing it for

$$P_n(x_1, x_2, x_3) = (x_1 + ix_2)^n \,.$$

• On $\mathbb{T}^2$, we have local well-posedness for Sobolev regularity ε-close to the typical regularity of simples of the Gibbs measure. This was used by Bourgain in his 1994 work in which he proved the existence and the uniqueness of the Gibbs measure dynamics.

• On a general manifold, the well-posedness theory is a 1/2 derivative away from the Gibbs measure. This is too much for our present understanding.

• On the sphere S^2, despite the 1/4 derivative gap, we succeed to build probabilistic well-posedness theory at the Gibbs measure regularity.

Theorem 5 (Burq-Camps-Chenmin Sun-Tz. 2024)
Let $\alpha > 1$. Then there exists a set Σ of full probability such that for every $\omega \in \Sigma$ there exists $T > 0$ such that the sequence $(u_N)_{N\geq 1}$ of solutions of NLS defined by

$$(i\partial_t + \Delta)u_N = |u_N|^2 u_N, \quad u(0,x) = \sum_{n=0}^{N} \frac{g_n(\omega)}{\langle \lambda_n \rangle^\alpha} \varphi_n(x) \qquad (6)$$

converge in $C([0,T]; L^2(S^2))$ to a distributional solution of NLS. More precisely, for every $T \in (0,1)$ there exists an event Σ_T such that

$$p(\Sigma_T) \geq 1 - C \exp(-c/T^\kappa)$$

for some positive C, c, κ and such that for every $\omega \in \Sigma_T$ the solutions of (6) converge in $C([0,T]; L^2(S^2))$ to a distributional solution of NLS.

The key stochastic object

• It is convenient to work with the Wick ordered NLS

$$(i\partial_t + \Delta)u = \mathcal{N}(u),$$

where

$$\mathcal{N}(u) = |u|^2 u - 2\|u\|_{L^2}^2\, u,$$

associated with the from

$$\mathcal{N}(u, v, w) = u\bar{v}w - 2\Big(\int u\bar{v}\Big)w.$$

• The free wave with datum $\psi_\alpha(x, \omega)$ is defined by

$$v_\alpha(t, x, \omega) = e^{it\Delta}(\psi_\alpha(x, \omega))$$

• A central object in this analysis is

$$I_\alpha(t, x, \omega) = \int_0^t e^{i(t-\tau)\Delta}\Big(\mathcal{N}(v_\alpha(\tau, x, \omega))\Big)d\tau$$

which represents the first nonlinear object which shows up when solving NLS with data $\psi_\alpha(x, \omega)$ by the Picard iteration scheme.

Regularity of the key stochastic object

Proposition 6 (Bourgain 1993)
Let $M=\mathbb{T}^2$. *Then for every* $\varepsilon>0$ *and* $t\in\mathbb{R}$,

$$I_\alpha(t,x,\omega)\in H^{\alpha-1+\frac{1}{2}-\varepsilon}(\mathbb{T}^2),\quad \text{almost surely.}$$

Proposition 7 (Burq-Camps-Latocca-Chenmin Sun-Tz. 2022)
Let $M=S^2$. *Then for every* $t\in\mathbb{R}$,

$$I_\alpha(t,x,\omega)\notin H^{\alpha-1}(S^2),\quad \text{almost surely.}$$

• **Question :** What is the regularity of the stochastic object in the case of hyperbolic geometry ? More generally, how the optimal regularity depends on the geometry of M ?
• In the case of an arbitrary manifold, we have (see e.g. Oh-Robert-Tz.)

$$I_\alpha(t,x,\omega)\in H^{\alpha-1-\varepsilon}(\mathbb{T}^2),\quad \text{almost surely}$$

which is non trivial because at least the typical regularity of the data is preserved.

Identifying the singular part of the nonlinearity on S^2

- Set

$$\Lambda(u) = \int_0^t e^{i(t-\tau)\Delta}\Big(\mathcal{N}(u(\tau))\Big)d\tau.$$

- If we denote by π_n the projector on spherical harmonics of degree n then we can write

$$\Lambda(u) = \sum_{n,n_1,n_2,n_3} c(n,n_1,n_2,n_3)\,\pi_n\Big(\pi_{n_1}u\,\pi_{n_2}\overline{u}\,\pi_{n_3}u\Big).$$

- Set

$$\tilde{\Lambda}(u) = \sum_{\substack{n\neq n_1,n_3\\ n_1\neq n_2, n_2\neq n_3}} c(n,n_1,n_2,n_3)\,\pi_n\Big(\pi_{n_1}u\,\pi_{n_2}\overline{u}\,\pi_{n_3}u\Big).$$

We have that

$$\tilde{\Lambda}(v_\alpha(t,x,\omega)) \in H^{\alpha-1+\frac{1}{2}-\varepsilon}(S^2), \quad \text{almost surely.}$$

Where the regularization comes from ?

• The resonant manifold associated with NLS on S^2 is

$$\{(n, n_1, n_2, n_3) \in \mathbb{N}^4 \, : \, n^2 - n_1^2 + n_2^2 - n_3^2 = 0\}.$$

• In the regime $n \neq n_1$ we can evoke the divisor bound :

$\forall \, \varepsilon > 0, \exists \, C, \forall \tau \in \mathbb{Z}, \forall N \geq 1,$ the number of integers n_1, n_2 satisfying

$$\tau = n_1^2 - n_2^2, \quad N \leq n_1, n_2 \leq 2N$$

is bounded by CN^{ε}.

• This gains essentially 1/2 derivatives compared to the basic estimate for the stochastic object (as used in singular SPDE) via the saving of one summations.

• The above divisor bound degenerates for $\tau = 0$. This degeneracy dictates the choice of the resolution ansatz and is responsible for the complications in the analysis.

The resolution ansatz

- Consider

$$i\partial_t u_N + \Delta u_N = \mathcal{N}(u_N), \quad u_N|_{t=0} = \Pi_N(\psi_\alpha(x,\omega)),$$

where

$$\Pi_N = \sum_{n \le N} \pi_n.$$

- We define the singular part of the nonlinearity by

$$\mathcal{N}_{(0,1)}(f,g,h) = \sum_{n,n_2,n_3} \pi_n\Big(\pi_n f \cdot \big(\overline{\pi_{n_2} g}\pi_{n_3} h - \int_{S^2} \overline{\pi_{n_2} g}\pi_{n_3} h\big)\Big).$$

- Following Bringmann and Deng-Nahmod-Yue, we split u_N as :

$$u_N = \sum_{M \le N} v_M, \quad v_M := u_M - u_{\frac{M}{2}}.$$

The resolution ansatz (sequel)

• Furthermore, we decompose

$$v_M := \psi_M + w_M,$$

anticipating that w_M is more regular and imposing that ψ_M solves the linear equation

$$(i\partial_t + \Delta)\psi_M = 2\Pi_M \mathcal{N}_{(0,1)}(\psi_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}), \quad \psi_M|_{t=0} = \mathbf{P}_M(\psi_\alpha(x,\omega)),$$

where $\mathbf{P}_M = \Pi_M - \Pi_{\frac{M}{2}}$.

• The equation for w_M (with 0 initial datum) reads:

$$(i\partial_t + \Delta)w_M = 2\Big(\mathcal{N}(v_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}) - \mathcal{N}_{(0,1)}(\psi_M, u_{\frac{M}{2}}, u_{\frac{M}{2}})\Big) + \mathcal{N}(u_{\frac{M}{2}}, v_M, u_{\frac{M}{2}}) + 2\mathcal{N}(v_M, v_M, u_{\frac{M}{2}}) + \mathcal{N}(v_M, u_{\frac{M}{2}}, v_M) + \mathcal{N}(v_M).$$

• We see that our ansatz removes the most singular high-low-low type interaction from the equation solved by the remainder w_M, i.e. there is a cancellation in

$$\mathcal{N}(v_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}) - \mathcal{N}_{(0,1)}(\psi_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}).$$

On the convergence

• For $\alpha > 1$, in the convergence proof, we crucially use that ψ_N and its renormalized square are bounded in L^∞ with some smallness. This leads to the study of the convergence of sequences $x_N(t)$ satisfying

$$\dot{x}_N(t) \leq C x_N(t), \quad x_N(0) \leq N^{-\delta},\ \delta > 0$$

which is straightforward.

• In the case $\alpha = 1$, ψ_N is no longer in L^∞, uniformly in N. This leads to serious complications because naive estimates on the operators involved in the analysis give only polynomial in N bounds (or $\log(N)^C$ with C large enough bigger than 1).

• However by using an iteration of the $TT^\star$ method (as sometimes done in random matrix problems), we arrive at logarithmic bounds which in turn leads to the study of the convergence of sequences $x_N(t)$ satisfying

$$\dot{x}_N(t) \leq C \log(N) x_N(t), \quad x_N(0) \leq N^{-\delta},\ \delta > 0$$

which still implies the convergence of $x_N(t)$, at least for small t depending on C and δ.

A remark

- Similar complications involving $\log(N)$ losses were addressed in closely related situations by Burq-Tz., Bourgain-Bulut, Chenmin Sun-Weijun Xu-Tz.

Theorem 8 (Burq-Camps-Chenmin Sun-Tz. 2025)
Let M be the standard $2d$ sphere. There exists a sequence $(c_N(\omega))_{N\geq 1}$ of positive numbers which diverges almost surely such that the sequence $(u_N)_{N\geq 1}$ defined by

$$(i\partial_t + \Delta + c_N)u_N = \Pi_N(|u_N|^2 u_N), \quad u(0,x) = \sum_{n=0}^{N} \frac{g_n(\omega)}{\langle \lambda_n \rangle} \varphi_n(x)$$

converges in $C(\mathbb{R}; H^{-s}(S^2))$, $s > 0$. The law of the limit at time t is given by the Gibbs measure associated with NLS on S^2.

- In the analysis we need a new proof of the 2002 result by Burq-Gérard-Tz. adapted to the [resonant]/[non resonant] decomposition of the nonlinearity.
- We crucially exploit the concentration of measure phenomenon applied to the analysis of random functions on S^2.
- In the iterated $TT^\star$ method, we exploit involved decorrelations in the S^2 variable related to the lack of translation invariance of the problem.
- Fortunately, we incorporate a relatively tiny part of the nonlinearity in the resolution ansatz.

Perspectives and open problems

• One may hope to remove the terms causing the log(N) losses by gauge transforms (in the spirit of a work by Tao or Oh-Wang-Tz.). We execute this idea in an ongoing work aiming to improve the result by Bourgain-Bulut for the radial NLS on B_3.

• Prove quasi-invariance of the gaussian measures induced by $\psi_\alpha(x, \omega)$ at least for α large enough ?

• Globalize the local solutions for $\alpha > 0$?

• What about other manifolds ?

Thank you for your attention !