Probabilistic well-posedness and Gibbs measure evolution for the non linear Schrödinger equation on the two dimensional sphere

NICOLAS BURQ
NICOLAS CAMPS
CHENMIN SUN
NIKOLAY TZVETKOV

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1. The Brézis-Gallouêt theorem and a motivation for the low regularity well-posedness theory of NLS

Let (M, g) be a compact $2d$ Riemannian manifold such that $\partial M = \emptyset$. The non linear Schrödinger equation (with a defocusing interaction) or NLS on M writes as follows :

$$(i\partial_t + \Delta)u = |u|^2 u. \tag{1.1}$$

If $u \in C^\infty(\mathbb{R} \times M)$ solves (1.1) then

$$E(u) = \|u\|^2_{H^1(M)} + \frac{1}{2}\int_M |u|^4$$

is a conserved quantity. Here, for $s \in \mathbb{R}$, $H^s(M)$ denotes the Sobolev space endowed with the norm

$$\|u\|_{H^s(M)} = \|(1-\Delta)^{s/2}(u)\|_{L^2(M)}\,.$$

We can show that (1.1) is locally well-posed in $H^s(M)$, $s > 1$ which is not sufficient for extending the solutions globally in time because $E(u)$ only controls the $H^1(M)$ norm of the solutions. On the other hand since M is two dimensional we have that $E(u)$ almost controls the $L^\infty(M)$ norm. More precisely we have that for every $s > 1$ there exists a positive constant C such that for every $u \in H^s(M)$ one has

$$\|u\|_{L^\infty(M)} \le C\|u\|_{H^1(M)}\Big(\log\Big(2 + \frac{\|u\|_{H^s(M)}}{\|u\|_{H^1(M)}}\Big)\Big)^{\frac{1}{2}}\,. \tag{1.2}$$

In addition, using some basic product estimates in Sobolev spaces, we can obtain that the solutions of (1.1) satisfy the energy estimate

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{H^s(M)} \le C\|u(t,\cdot)\|^2_{H^s(M)}\|u(t,\cdot)\|^2_{L^\infty(M)} \tag{1.3}$$

which in conjugation with (1.2) yields, using that the energy controls the H^1 norm,

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{H^s(M)} \le C\|u(t,\cdot)\|^2_{H^s(M)}\log\big(2 + \|u\|_{H^s(M)}\big)$$

and therefore after the integration of the last inequality we obtain that

$$\|u(t,\cdot)\|_{H^s(M)} \le e^{Ce^{C|t|}}$$

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which implies that the $H^s(M)$ of the solutions of (1.1) cannot blow-up in finite time. Therefore, we have the following statement.

Theorem 1 (Brézis-Gallouêt, [7])**.** *The Cauchy problem associated with* (1.1) *is globally well-posed in* $H^s(M)$*,* $s > 1$.

It was shown in [12] that the above result can be extended to the three dimensional case. This extension requires a delicate use of the dispersive properties of the Schrödinger propagator $\exp(it\Delta)$. We will not discuss these three dimensional extensions here but these dispersive properties in dimension 2 will both be a starting point and important argument in our analysis; see Section 2 below.

The main topic in this exposé will be how much we can extend the result of Theorem 1 to lower regularity spaces $H^s(M)$ for some $s < 1$. One may naturally demand why we ask this question. There are several possible motivations. We present below the one coming from the Gibbs measure associated with (1.1).

At a non rigorous level, the formal object

$$Z^{-1}\, e^{-H(u)}\, du = Z^{-1}\, e^{-\frac{1}{2}\int |u|^4}\, e^{-\|u\|^2_{H^1}}\, du \tag{1.4}$$

is the Gibbs measure associated with NLS.

Rigorously, after a renormalization (1.4) is defined as a measure absolutely continuous with respect to the measure defined by the map

$$\omega \longmapsto \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\langle \lambda_n \rangle}\, \varphi_n(x),$$

where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots,$

$$-\Delta \varphi_n = \lambda_n^2 \varphi_n,$$

$(\varphi_n)_{n\geq 0}$ being an orthonormal bases of $L^2(M)$ of eigenfunctions of Δ and $(g_n)_{n\geq 0}$ being a family of i.i.d. standard complex gaussians. Here, by definition $\langle \lambda_n \rangle = (1+\lambda_n^2)^{\frac{1}{2}}$.

We have that

$$\psi_\alpha(x,\omega) := \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\langle \lambda_n \rangle^\alpha}\, \varphi_n(x) \in H^s(M),\ s < \alpha - 1, \text{ almost surely.} \tag{1.5}$$

Moreover $\psi_\alpha \notin H^{\alpha-1}(M)$ almost surely (see e.g. [15]).

Let us give the proof of (1.5). By definition, we can write

$$\int_\Omega |(1-\Delta)^{s/2}\, \psi_\alpha(x,\omega)|^2 dp(\omega) = \sum_{n=0}^{\infty} \frac{|\varphi_n(x)|^2}{\langle \lambda_n \rangle^{2\alpha-2s}}$$

which can be estimated by

$$\sum_{N-\text{dyadic}} N^{2s-2\alpha} \sum_{\lambda_n \sim N} |\varphi_n(x)|^2 \lesssim \sum_{N-\text{dyadic}} N^{2s-2\alpha+2} < \infty$$

because thanks to the Hörmander spectral function theorem [21] we have that

$$\sum_{\lambda_n \sim N} |\varphi_n(x)|^2 \lesssim N^2,$$

uniformly in $x \in M$ (N^2 becomes N^d in dimension d). This ends the proof of (1.5).

Therefore, if one wishes to solve NLS with initial data typical with respect to the Gibbs measure then one needs to show well-posedness for a large dense set of initial data in

$$\bigcap_{s<0} H^s(M) \cap (L^2(M))^c \quad \text{(the previous analysis with } \alpha = 1\text{)}.$$

In this Gibbs measure problem the main issue is the local well-posdness because thanks to work by Bourgain [2] we know that relatively quantified probabilistic local well-posedness with respect to the Gibbs measure implies the global well-posedness by exploiting measure propagation arguments instead of conservation laws.

2. **Review of the deterministic well-posedness**

Concerning the *local* well-posedness, we have the following notable improvement of Theorem 1.

Theorem 2 ([12]). *NLS* (1.1) *is locally well-posed in* $H^s(M)$*,* $s > 1/2$*. For every datum of size* $R \geq 1$ *in* $H^s(M)$ *the solution is defined on* $[0, T]$ *with* $T \approx R^{-\alpha}$*,* $\alpha > 0$*.*

As mentioned above, if we only use the energy estimate (1.3) then we would have the previous result only for $s > 1$. The main ingredient in the proof of Theorem 2 is the Strichartz estimate

$$\|\exp(it\Delta)(f)\|_{L^2([0,1];L^\infty(M))} \leq C\|f\|_{H^s(M)}, \quad s > 1/2. \tag{2.1}$$

Estimate (2.1) fails for $s < 1/2$ in the case of S^2. Once we have (2.1) we can solve (1.1) locally in time on the claimed time interval by performing a fixed point argument in the space

$$X_T = L^{2-\delta}([0, T]; L^\infty(M)) \cap L^\infty([0, T]; H^s(M)), \quad s > 1/2,$$

where $\delta > 0$ is sufficiently small (depending of s). If one wishes to improve Theorem 2 then one should find an argument which does not aim to control the quantity $\|u\|_{L^{2-\delta}([0,T];L^\infty(M))}$. This quantity is the analogue of $\|\nabla u\|_{L^1([0,T];L^\infty(M))}$ appearing in some fluid dynamics PDE such as the incompressible Euler equation. Such an argument was found in the seminal work [1] in the case when M is the flat torus. This led to the following statement.

Theorem 3 (Bourgain, [1]). *NLS is locally well-posed in* $H^s(\mathbb{T}^2)$*,* $s > 0$*, where* $\mathbb{T}^2$ *denotes the flat torus. For every datum of size* $R \geq 1$ *in* $H^s(\mathbb{T}^2)$ *the solution is defined on* $[0, T]$ *with* $T \approx R^{-\alpha}$*,* $\alpha > 0$*.*

The main ingredient in the proof of Theorem 3 is the estimate

$$\|P_Q \exp(it\Delta)(f)\|_{L^4([0,1]\times\mathbb{T}^2)} \leq C_\varepsilon \, |Q|^\varepsilon \, \|f\|_{L^2(\mathbb{T}^2)}, \quad \varepsilon > 0, \tag{2.2}$$

where $Q \subset \mathbb{R}^2$ is a square and P_Q is the projector

$$P_Q(f) = \sum_{n \in \mathbb{Z}^2 \cap Q} \hat{f}(n) \, e^{in\cdot x} \, .$$

Once estimate (2.2) is established one can solve (1.1) by a fixed point argument in the $X^{s,b}$ spaces introduced by Bourgain in [1]. We refer to the article by Ginibre [19] which contains a pedagogical presentation of the ideas of [1]. Even if [19] is considered as an expository work it is in our opinion a truly original contribution to the subject which strongly influenced many subsequent contributions. An important advantage of the use of the Bourgain spaces is that they are sensitive to the fact that the quantity in the right hand-side of (2.2) only depends on the size of the square Q and *not* on its position in the frequency space. In the remarkable recent work [20], Herr-Kwak extended the solution obtained in Theorem 3 globally in time.

The technique introduced in [1] is quite restricted to the nature of the flat torus and its extension to a general manifold is very much an open problem. There is however a notable exception. More precisely, the technique of [1] was extended to the case of the standard sphere as shown by the following statement.

Theorem 4 ([13, 14]). *NLS (1.1) is locally well-posed in $H^s(S^2)$, $s > 1/4$. For every datum of size $R \geq 1$ in $H^s(S^2)$ the solution is defined on $[0,T]$ with $T \approx R^{-\alpha}$, $\alpha > 0$. Moreover, for $s < 1/4$ NLS on S^2 fails to be semi-linearly well-posed.*

The main ingredients in the proof of Theorem 4 are arithmetic properties of the sequence $(n^2)_{n\in\mathbb{Z}}$ and the estimate

$$\|P_n P_m\|_{L^2(S^2)} \leq C(1+\min(n,m))^{1/4}\|P_n\|_{L^2(S^2)}\|P_m\|_{L^2(S^2)} \tag{2.3}$$

where P_n and P_m are spherical harmonics of degrees n and m respectively and C is an absolute constant (we do not know the optimal one even for $m = n$). For $s < 1/4$, we use that estimate (2.3) is asymptotically optimal by testing it on the spherical harmonics

$$P_n(x_1, x_2, x_3) = (x_1 + ix_2)^n \,.$$

By a failure of semi-linear well-posedness we mean that if the problem is well-posed for $s < 1/4$ then the flow map cannot be locally Lipschitz continuous on $H^s(S^2)$ while for $s > 1/4$ this property holds as a byproduct of the proof of Theorem 4.

3. Probabilistic well-posedness

Let us now discuss the results presented in the previous section in the context of initial data distributed according to the Gibbs measure. On $\mathbb{T}^2$, we have local well-posedness for Sobolev regularity ε-close to the typical regularity of samples of the Gibbs measure. This was used by Bourgain in [3], a work in which he proved the existence and the uniqueness of the Gibbs measure dynamics. The article [3] was at the origin of many developments in what is now called probabilistic well-posedness of nonlinear dispersive PDE. We do not aim to survey these developments here. We will only restrict to the results in connection with the Gibbs measure problem for (1.1).

In sharp contrast with the flat torus, on a general manifold, the well-posedness theory of Theorem 2 is a 1/2 derivative away from the Gibbs measure. This is too much for our present understanding.

However, on the sphere S^2, despite the 1/4 derivative gap between the well-posedness regularity restriction of Theorem 4 and the Gibbs measure regularity we succeeded to build probabilistic well-posedness theory at the level of the Gibbs measure regularity. As a first step in this direction we established the following probabilistic local well-posedness result for data ϵ close to the Gibbs measure regularity.

Theorem 5 ([9]). *Let M be the standard 2d sphere. Let $\alpha > 1$. Then there exists a set Σ of full probability such that for every $\omega \in \Sigma$ there exists $T > 0$ such that the sequence $(u_N)_{N\geq 1}$ of solutions of NLS defined[1] by*

$$(i\partial_t + \Delta)u_N = |u_N|^2 u_N, \quad u(0,x) = \sum_{n=0}^{N} \frac{g_n(\omega)}{\langle \lambda_n \rangle^\alpha}\, \varphi_n(x) \tag{3.1}$$

[1]The existence of u_N is ensured by Theorem 1.

converges in $C([0,T];L^2(S^2))$ *to a distributional solution of NLS.*
More precisely, for every $T\in(0,1)$ *there exists an event* Σ_T *such that*

$$p(\Sigma_T)\geq 1-C\exp(-c/T^\kappa)$$

for some positive C,c,κ *and such that for every* $\omega\in\Sigma_T$ *the solutions of* (3.1) *converge in* $C([0,T];L^2(S^2))$ *to a distributional solution of NLS.*

Observe that the second part of the last statement is a quantitative version of the first part and we may take $\Sigma=\bigcup_{n=1}^{\infty}\Sigma_{1/n}$.

3.1. **The key stochastic object.**

In this section we discuss regularity properties of the key stochastic object which appears in the analysis of the NLS (1.1) with random initial data. It is convenient to work with the Wick ordered NLS

$$(i\partial_t+\Delta)u=\mathcal{N}(u), \tag{3.2}$$

where

$$\mathcal{N}(u)=|u|^2u-2\|u\|_{L^2}^2\,u,$$

associated with the form

$$\mathcal{N}(u,v,w)=u\overline{v}w-2\Big(\int u\overline{v}\Big)w.$$

Using the L^2 conservation law, we obtain that, at least formally, u solves (3.2) if and only if $e^{iKt}u$ solves (1.1) where $K\in\mathbb{R}$ is given in terms of the initial datum by

$$K=-2\|u(0,\cdot)\|_{L^2(M)}^2\,.$$

Therefore, at least for L^2 solutions, the passage from (3.2) to (1.1) is straightforward.

Recall that $\psi_\alpha(x,\omega)$ is defined by (1.5). Then the free wave with datum $\psi_\alpha(x,\omega)$ is defined by

$$v_\alpha(t,x,\omega)=e^{it\Delta}(\psi_\alpha(x,\omega)).$$

A central object in this analysis is

$$I_\alpha(t,x,\omega)=\int_0^t e^{i(t-\tau)\Delta}\Big(\mathcal{N}(v_\alpha(\tau,x,\omega))\Big)d\tau$$

which represents the first nonlinear object which shows up when solving NLS with data $\psi_\alpha(x,\omega)$ by the Picard iteration scheme.

Let us now turn to the regularity properties of the key stochastic object $I_\alpha(t,x,\omega)$. Using the analysis of [22] one can show that in the case of an arbitrary manifold, if $\alpha\geq 1$ then for every $\varepsilon>0$ we have

$$I_\alpha(t,x,\omega)\in H^{\alpha-1-\varepsilon}(M),\quad \text{almost surely} \tag{3.3}$$

which is non trivial because at least the typical regularity of the data is preserved (recall (1.5)).

It was shown by Bourgain in [3] that in the case of the flat torus one can strongly improve on (3.3). More precisely the following remarkable statement holds.

Proposition 6. *Let* $M=\mathbb{T}^2$. *Suppose that* $\alpha\geq 1$. *Then for every* $\varepsilon>0$ *and* $t\in\mathbb{R}$,

$$I_\alpha(t,x,\omega)\in H^{\alpha-1+\frac{1}{2}-\varepsilon}(\mathbb{T}^2),\quad \text{almost surely.}$$

In other words, in the case of the flat torus one has an (almost) 1/2 derivative improvement on the general statement (3.3).

On the other hand, it was show by Latocca and the authors in [8] that in the case of the standard sphere there is *no improvement* on (3.3). More precisely the following statement holds.

Proposition 7. *Let $M = S^2$. Suppose that $\alpha \geq 1$. Then for every $t \in \mathbb{R}$,*

$$I_\alpha(t, x, \omega) \notin H^{\alpha-1}(S^2), \quad \text{almost surely.}$$

In [8] it was only shown the divergence of the variance of $I_\alpha(t, x, \omega)$. We believe that it may be extended to an almost sure divergence, as stated in Proposition 7, by an application of a multi-linear version of the Fernique integrability theorem [18].

It would be interesting to decide what is the regularity of the stochastic object in the case of hyperbolic geometry. More generally, how the optimal regularity depends on the geometry of M is a largely open problem.

3.2. Identifying the singular part of the nonlinearity on S^2

It turns out that the lack of regularity of $I_\alpha(t, x, \omega)$ on S^2 displayed by Proposition 7 is because of some specific interactions. These interactions are very similar to those which lead to the failure of semi-linear well-posedness in Theorem 4. They are related to the existence of stable geodesics on S^2 (absent on the flat torus or on hyperbolic manifolds). These geodesics allow to observe a concentration on a closed geodesics in $I_\alpha(t, x, \omega)$ which leads to Proposition 7. However, relatively few interactions lead to such concentrations. Amazingly, we arrive to measure the contributions to $I_\alpha(t, x, \omega)$ leading to concentrations on a stable geodesics in terms of arithmetic properties of the so called resonant function associated with the cubic nonlinear interaction.

Let us now describe more precisely the singular part of the nonlinearity. Set

$$\Lambda(u) = \int_0^t e^{i(t-\tau)\Delta}\Big(\mathcal{N}(u(\tau))\Big)d\tau.$$

If we denote by π_n the projector on spherical harmonics of degree n then we can write

$$\Lambda(u) = \sum_{n,n_1,n_2,n_3} c(t, n, n_1, n_2, n_3)\, \pi_n\big(\pi_{n_1} u\, \pi_{n_2} \overline{u}\, \pi_{n_3} u\big).$$

Set

$$\tilde{\Lambda}(u) = \sum_{\substack{n \neq n_1, n_3 \\ n_1 \neq n_2, n_2 \neq n_3}} c(t, n, n_1, n_2, n_3)\, \pi_n\big(\pi_{n_1} u\, \pi_{n_2} \overline{u}\, \pi_{n_3} u\big).$$

We have that

$$\tilde{\Lambda}(v_\alpha(t, x, \omega)) \in H^{\alpha-1+\frac{1}{2}-\varepsilon}(S^2), \quad \text{almost surely.}$$

In other words if we ignore the "diagonal" contributions of n, n_1, n_2, n_3 to $\Lambda(u)$ then the regularity strongly improves.

Let us next explain where such a regularization comes from. The resonant manifold associated with NLS on S^2 is

$$\{(n, n_1, n_2, n_3) \in \mathbb{N}^4 \;:\; n^2 - n_1^2 + n_2^2 - n_3^2 = 0\}.$$

On this manifold the nonlinear effects have the strongest strength. In the regime $n \neq n_1$ we can evoke the divisor bound :

$$\forall \varepsilon > 0, \quad \exists C, \quad \forall \tau \in \mathbb{Z}\backslash\{0\}, \quad \forall N \geq 1, \quad \text{the number of integers } n_1, n_2 \text{ satisfying}$$
$$\tau = n_1^2 - n_2^2, \quad N \leq n_1, n_2 \leq 2N$$

is bounded by CN^{ε}. This gains essentially 1/2 derivatives compared to the basic estimate for the stochastic object given by (3.3).

The above divisor bound degenerates for $\tau = 0$. This degeneracy determines the singular part of the nonlinearity. It also dictates the choice of the resolution ansatz and it is responsible for the complications in the analysis.

3.3. **The resolution ansatz**

Consider

$$i\partial_t u_N + \Delta u_N = \mathcal{N}(u_N), \quad u_N|_{t=0} = \Pi_N(\psi_\alpha(x,\omega)),$$

where

$$\Pi_N = \sum_{n \leq N} \pi_n.$$

We define the singular part of the nonlinearity by

$$\mathcal{N}_{(0,1)}(f,g,h) = \sum_{n,n_2,n_3} \pi_n\Big(\pi_n f \cdot \big(\overline{\pi_{n_2} g}\pi_{n_3} h - \int_{S^2} \overline{\pi_{n_2} g}\pi_{n_3} h\big)\Big).$$

Following Bringmann [6], we split u_N as follows:

$$u_N = \sum_{M \leq N} v_M, \quad v_M := u_M - u_{\frac{M}{2}}.$$

Furthermore, following Deng-Nahmod-Yue [17] we decompose

$$v_M := \psi_M + w_M,$$

anticipating that w_M is more regular and imposing that ψ_M solves the linear equation

$$(i\partial_t + \Delta)\psi_M = 2\Pi_M \mathcal{N}_{(0,1)}(\psi_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}), \quad \psi_M|_{t=0} = \mathbf{P}_M(\psi_\alpha(x,\omega)),$$

where $\mathbf{P}_M = \Pi_M - \Pi_{\frac{M}{2}}$.

The equation for w_M (with vanishing initial datum) reads:

$$(i\partial_t + \Delta)w_M = 2\big(\mathcal{N}(v_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}) - \mathcal{N}_{(0,1)}(\psi_M, u_{\frac{M}{2}}, u_{\frac{M}{2}})\big) +$$
$$\mathcal{N}(u_{\frac{M}{2}}, v_M, u_{\frac{M}{2}}) + 2\mathcal{N}(v_M, v_M, u_{\frac{M}{2}}) + \mathcal{N}(v_M, u_{\frac{M}{2}}, v_M) + \mathcal{N}(v_M).$$

We see that our ansatz removes the most singular high-low-low type interaction from the equation solved by the remainder w_M, i.e. there is a cancellation in

$$\mathcal{N}(v_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}) - \mathcal{N}_{(0,1)}(\psi_M, u_{\frac{M}{2}}, u_{\frac{M}{2}}).$$

The randomness contained in $\psi_\alpha(x,\omega)$ is exploited inductively on M. At each scale M, we exploit the randomness in $\mathbf{P}_M(\psi_\alpha(x,\omega))$ and the exceptional sets coming from the randomness in $\mathbf{P}_{M_0}(\psi_\alpha(x,\omega))$, $M_0 < M$ are no longer modified.

For $\alpha > 1$, in the convergence proof, we crucially use that ψ_N and its renormalized square are almost surely bounded in L^∞ with some smallness. This leads to the study of the convergence of sequences $x_N(t)$ satisfying

$$\dot{x}_N(t) \leq C x_N(t), \quad x_N(0) \leq N^{-\delta}, \ \delta > 0$$

which is straightforward.

3.4. Gibbs measure initial data

In the case $\alpha = 1$, ψ_N is no longer in L^∞, uniformly in N. This leads to serious complications because naive estimates on the operators involved in the analysis give only polynomial in N bounds (or $\log(N)^C$ with C large enough bigger than 1). However by using an iteration of the $TT^\star$ method (as sometimes done in random matrix problems), we arrive at logarithmic bounds which in turn leads to the study of the convergence of sequences $x_N(t)$ satisfying

$$\dot{x}_N(t) \leq C \log(N) x_N(t), \quad x_N(0) \leq N^{-\delta}, \ \delta > 0$$

which still implies the convergence of $x_N(t)$, at least for small t depending on C and δ. It is worth noticing that similar complications involving $\log(N)$ losses were addressed in closely related situations in [4, 16, 5, 23].

In the case $\alpha = 1$, we have the following statement.

Theorem 8 ([10])**.** *Let M be the standard 2d sphere. There exists a sequence $(c_N(\omega))_{N\geq 1}$ of positive numbers which diverges almost surely such that the sequence $(u_N)_{N\geq 1}$ defined by*

$$(i\partial_t + \Delta + c_N(\omega))u_N = \Pi_N(|u_N|^2 u_N), \quad u(0,x) = \sum_{n=0}^{N} \frac{g_n(\omega)}{\langle \lambda_n \rangle} \varphi_n(x)$$

converges in $C(\mathbb{R}; H^{-s}(S^2))$, $s > 0$ (endowed with the compact-open topology). For every $t \in \mathbb{R}$ the Gibbs measure associated with NLS on S^2 is invariant under the map which associates to the initial data the solution at time t.

We observe that compared to Theorem 5 in Theorem 8 the renormalization constants $c_N(\omega)$ appear which is a usual feature in this field. Moreover the convergence in Theorem 8 is global in time thanks to measure invariance considerations.

In the analysis leading to the proofs of Theorem 5 and Theorem 8 we need a new proof of Theorem 4 adapted to the [resonant]/[non resonant] decomposition of the nonlinearity.

In the analysis we crucially exploit the concentration of measure phenomenon applied to the analysis of random functions on S^2. In an iterated $TT^\star$ method, we exploit involved decorrelations in the S^2 variable related to the lack of translation invariance of the problem.

Let us finally mention that fortunately, we incorporate a relatively tiny part of the nonlinearity in the resolution ansatz. This will certainly not be sufficient in the case of a stronger nonlinear interaction.

3.5. Perspectives and open problems

One may hope to remove the terms causing the $\log(N)$ losses by gauge transforms. We execute this idea in an ongoing work [11] aiming to improve the result by Bourgain-Bulut [5] for the radial NLS on the unit ball of $\mathbb{R}^3$.

In the context of Theorem 5, it would be very interesting to develop a method which allows to extend the obtained solutions globally in time. For $\alpha > 1$ no invariant measure is available and the globalization problem is a challenging one. Moreover, it would be interesting to study the quasi-gaussianity of the law of the solutions, at least for α large enough.

We end this short exposé by recalling that the results of Theorem 5 and Theorem 8 for more general manifolds are completely open problems and that the negatively curvature manifolds are a particularly appealing case.

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Nicolas Burq
Université Paris-Saclay, Laboratoire de Mathématique d'Orsay, UMR CNRS 8628, Orsay, France, and Institut universitaire de France
nicolas.burq@universite-paris-saclay.fr

Nicolas Camps
Univ Rennes, IRMAR - UMR CNRS 6625, F-35000 Rennes, France
nicolas.camps@univ-rennes.fr

Chenmin Sun
CNRS, Université Paris-Est Créteil, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR CNRS 8050, Créteil, France
chenmin.sun@cnrs.fr

Nikolay Tzvetkov
Ecole Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliqués, UMR CNRS 5669, Lyon, France
nikolay.tzvetkov@ens-lyon.fr