February 4, 2021

On the macroscopical description of the flow of nonlinear wave equations

Nikolay Tzvetkov

CY Cergy-Paris Université

$\boxed{\text{Sobolev spaces}}$

- For a function f on $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ given by its Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

we define the Sobolev norm H^s of f as

$$\|f\|_{H^s}^2 = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s}\, |\widehat{f}(n)|^2\,.$$

Here $\widehat{f}(n)$ are the Fourier coefficients of f and

$$\langle n\rangle = (1 + n_1^2 + \cdots + n_d^2)^{\frac{1}{2}}\,.$$

- For $s \geq 0$ an integer, we have

$$\|f\|_{H^s} \approx \sum_{|\alpha|\leq s} \|\partial^\alpha f\|_{L^2}\,.$$

The nonlinear Klein-Gordon equation

- Consider

$$(\partial_t^2 - \Delta + 1)u + u^3 = 0, \tag{1}$$

where Δ is the Laplacian and u is a real valued function.

- If we multiply the equation (1) by $\partial_t u$, we formally get

$$\frac{d}{dt}\Big(\int \Big(|\nabla u|^2 + u^2 + (\partial_t u)^2 + \frac{1}{2}u^4\Big)\Big) = 0 \,.$$

Therefore $(u, \partial_t u) \in H^1 \times L^2$ is a natural framework for the solutions of (1).

The nonlinear Klein-Gordon equation

Theorem 1 (classical)

• *For every* $(u_0, u_1) \in H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ *there exists a unique global solution of*

$$(\partial_t^2 - \Delta + 1)u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x)$$

in the class $(u, \partial_t u) \in C(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$.

• *If in addition* $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ *for some* $s \geq 1$ *then*

$$(u, \partial_t u) \in C(\mathbb{R}; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)).$$

The dependence with respect to the initial data is continuous.

• The local in time part of Theorem 1 can be extended to the case $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$, $s \geq 1/2$, and the global in time part to $s > 3/4$ (Kenig-Ponce-Vega, Gallagher-Planchon, Roy).

• We conjecture that Theorem 1 remains true for $s \geq 1/2$ (related recent work by Dodson).

Theorem 2

Let $s \in (0, 1/2)$ *et* $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. *There exists a sequence*

$$u_N(t,x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \quad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta + 1)u_N + u_N^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N(0) - u_0, \partial_t u_N(0) - u_1)\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \sup_{0 \le t \le T} \|u_N(t)\|_{H^s(\mathbb{T}^3)} = +\infty.$$

• The proof is based on an idea introduced by **Lebeau** and further developed by Christ-Colliander-Tao, Burq-Tz., Xia.

$\boxed{\text{Solving the equation by probabilistic methods}}$

• Inspired by the work of **Bourgain in the early 1990's** on invariant measures for NLS, we can ask whether some form of well-posedness survives for initial data in

$$H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3), \quad s < 1/2. \tag{2}$$

• The answer of this question is positive if we endow the space (2) with a non degenerate probability measure such that we have the existence, the uniqueness, and a form of continuous dependence almost surely with respect to this measure.

Choice of the measure

• Fix a real number σ. We will choose the initial data among the realisations of the following random series

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}\,, \qquad u_1^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x}\,. \tag{3}$$

Here $\{g_n\}_{n\in\mathbb{Z}^3}$ et $\{h_n\}_{n\in\mathbb{Z}^3}$ are two families of independent random variables conditioned by $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$, so that u_0^ω and u_1^ω are real valued.

• In addition, we suppose that for $n \neq 0$, g_n and h_n are complex gaussians from $\mathcal{N}_{\mathbb{C}}(0,1)$, and that g_0 and h_0 are standard real gaussians from $\mathcal{N}(0,1)$.

• The initial data (3) belong almost surely to $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ for $s < \sigma - \frac{1}{2} (= \sigma + 1 - \frac{3}{2})$. Moreover, the probability of the event

$$(u_0^\omega, u_1^\omega) \in H^{\sigma-\frac{1}{2}}(\mathbb{T}^3) \times H^{\sigma-\frac{3}{2}}(\mathbb{T}^3)$$

is zero.

Reformulation of the ill-posedness result

Theorem 3

Let $\sigma \in (1/2, 1)$ and $0 < s < \sigma - 1/2$. For almost every ω, there exists a sequence

$$u_N^\omega(t,x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \qquad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta + 1)u_N^\omega + (u_N^\omega)^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N^\omega(0) - u_0^\omega, \partial_t u_N^\omega(0) - u_1^\omega)\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \sup_{0 \le t \le T} \|u_N^\omega(t)\|_{H^s(\mathbb{T}^3)} = +\infty.$$

We can however prove the following result:

Theorem 4 (Burq-Tz. (2010))
Let $\sigma \in (1/2, 1)$ and $0 < s < \sigma - 1/2$. Define (thanks to the classical well-posedness result) the sequence $(u_N)_{N\geq 1}$ of solutions of

$$(\partial_t^2 - \Delta + 1)u + u^3 = 0 \tag{4}$$

with C^∞ initial data

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}, \qquad u_1^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x}.$$

The sequence $(u_N)_{N\geq 1}$ converges almost surely as $N \to \infty$ in $C(\mathbb{R}; H^s(\mathbb{T}^3))$ to a (unique) limit u which satisfies (4) in the distributional sense.

- Therefore the type of the approximation of the initial data is crucial when we prove probabilistic low regularity well-posedness.
- Even if we consider the approximation of the initial data by Fourier truncation there is dense set of pathological data such that the statement of Theorem 4 does not hold (recent work by **Sun-Tz.**).
- We can prove uniqueness in a suitable functional framework.
- We can consider more general randomisations (this fact had an important impact in the field).

Going further

Theorem 5 (Oh-Pocovnicu-Tz. (2019))
Let $\sigma \in (\frac{1}{4}, \frac{1}{2}]$ and $s < \sigma - 1/2$. There exists a divergent sequence $(c_N)_{N\geq 1}$ such that if we denore by $(u_N^\omega)_{N\geq 1}$ the solution of

$$\partial_t^2 u - \Delta u + u - c_N u + u^3 = 0, \tag{5}$$

with initial data given by

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}\,, \qquad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x}$$

then for almost every ω there exists $T_\omega > 0$ such that $(u_N^\omega)_{N\geq 1}$ converges in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$.

- Theorem 5 was the first step in the study of the nonlinear wave equation in Sobolev spaces of negative indexes.
- Recent work by **Bringmann** allows to have $\sigma > 0$ (there are related works by **Gubinelli-Koch-Oh, Deng-Nahmod-Yue**).
- For $\sigma = 0$ there is exceptionally an invariant measure and one may hope to get global solutions by an argument of Bourgain. Solving this problem now only seems a question of time.

A puzzling remark

• Unfortunately, even if these last results are satisfactory as far individual trajectories are concerned, these results give no inside on the macroscopical description of the flow. Namely, we **do not know** what is the transport by the flow of the measure induced by the map

$$\omega \longmapsto \Big(\sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}, \quad \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x} \Big) .$$

• It turned out that the answer of the last question of macroscopical description is hard from being obvious even for large σ (gaussian fields with regular typical elements).

• However, as we shall see right now we find easily the answer of the above question for the linear equation.

$\boxed{\text{The linear equation}}$

- Consider the linear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \qquad (6)$$

where u_0 and u_1 are real valued and $u : \mathbb{R} \times \mathbb{T}^3 \longrightarrow \mathbb{R}$. The solutions of (6) are given by

$$S(t)(u_0, u_1) \equiv \cos(t\sqrt{1-\Delta})(u_0) + \frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}(u_1),$$

where

$$\cos(t\sqrt{1-\Delta})(u_0) \equiv \sum_{n\in\mathbb{Z}^3} \cos(t\langle n\rangle)\widehat{u_0}(n)\, e^{in\cdot x},$$

$$\frac{\sin(t\sqrt{1-\Delta})}{\sqrt{1-\Delta}}(u_1) \equiv \sum_{n\in\mathbb{Z}^3} \frac{\sin(t\langle n\rangle)}{\langle n\rangle}\widehat{u_1}(n)\, e^{in\cdot x}.$$

The free evolution

• It follows directly from the definition that the operator

$$\bar{S}(t) \equiv (S(t), \partial_t S(t)),$$

where

$$\partial_t S(t)(u_0, u_1) \equiv -\sqrt{1-\Delta}\,\sin(t\sqrt{1-\Delta})(u_0) + \cos(t\sqrt{1-\Delta})(u_1)$$

is bounded on $H^\sigma \times H^{\sigma-1}$, $\bar{S}(0) = \mathrm{Id}$ and $\bar{S}(t+\tau) = \bar{S}(t) \circ \bar{S}(\tau)$.

• In the proof of the boundedness of $\bar{S}(t)$ on $H^\sigma \times H^{\sigma-1}$, we only use the boundedness of $\cos(t\langle n\rangle)$ and $\sin(t\langle n\rangle)$. One may use the oscillations of $\cos(t\langle n\rangle)$ and $\sin(t\langle n\rangle)$ for $|n| \gg 1$ in order to get more involved L^p, $p > 2$ properties of the map $\bar{S}(t)$ (Strichartz estimates).

Invariant spaces of the linear evolution

Set

$$l_1 = \begin{pmatrix} \cos(n \cdot x) \\ 0 \end{pmatrix}, \; l_2 = \begin{pmatrix} 0 \\ \cos(n \cdot x) \end{pmatrix} .$$

Then by definition for real numbers λ_1, λ_2, we can write

$$\begin{aligned} \bar{S}(t)(\lambda_1 l_1 + \lambda_2 l_2) = (\lambda_1 \cos(t\langle n\rangle) + \lambda_2 \langle n\rangle^{-1} \sin(t\langle n\rangle))l_1 \\ + (-\lambda_1 \langle n\rangle \sin(t\langle n\rangle) + \lambda_2 \cos(t\langle n\rangle))l_2 \end{aligned}$$

Hence in the plane spanned by l_1, l_2, the map $\bar{S}(t)$ is represented by

$$A = \begin{pmatrix} \cos(t\langle n\rangle) & -\langle n\rangle \sin(t\langle n\rangle) \\ \langle n\rangle^{-1} \sin(t\langle n\rangle) & \cos(t\langle n\rangle) \end{pmatrix}.$$

We have that $\det(A) = 1$ and that for every σ, the quadratic form

$$Q(X, Y) = \langle n\rangle^{2\sigma+2} X^2 + \langle n\rangle^{2\sigma} Y^2$$

is preserved by $\bar{S}(t)$.

Invariant spaces of the linear evolution (sequel)

Let us equip the line spanned by l_1 with the gaussian measure

$$\frac{\langle n\rangle^{\sigma+1}}{\sqrt{2\pi}} e^{-\frac{\langle n\rangle^{2\sigma+2}x^2}{2}} dx,$$

the line spanned by l_2 with the gaussian measure

$$\frac{\langle n\rangle^{\sigma}}{\sqrt{2\pi}} e^{-\frac{\langle n\rangle^{2\sigma}x^2}{2}} dx.$$

Denote by γ the natural product measure in the plane spanned by l_1 and l_2. Then thanks to the previous discussion, we have :

Proposition 6

The measure γ is invariant under the restriction of $\bar{S}(t)$ to the plane spanned by l_1 and l_2.

A similar analysis holds concerning the plane spanned by

$$\begin{pmatrix} \sin(n\cdot x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sin(n\cdot x) \end{pmatrix} .$$

Invariance under the free evolution

Denote by μ_σ the measure induced by the map

$$\omega \longmapsto \Big(\sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}, \quad \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x} \Big) .$$

The previous analysis yields :

Proposition 7

Let $\sigma \in \mathbb{R}$. The measure μ_σ is invariant under the linear evolution $\bar{S}(t)$.

Question : How much this property survives for the nonlinear flow ?

The measure μ_σ and the nonlinear equation

- Consider again the nonlinear Klein-Gordon equation

$$\partial_t^2 u - \Delta u + u + u^3 = 0, \tag{7}$$

where $u : \mathbb{R} \times \mathbb{T}^3 \longrightarrow \mathbb{R}$.

- We rewrite (7) as the first order system

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3. \tag{8}$$

- One can rewrite (8) as a Hamiltonian system

$$\partial_t u = \frac{\delta E}{\delta v}, \quad \partial_t v = -\frac{\delta E}{\delta u},$$

where

$$E(u,v) = \frac{1}{2}\int_{\mathbb{T}^3} \left(u^2 + |\nabla u|^2 + v^2\right) + \frac{1}{4}\int_{\mathbb{T}^3} u^4 .$$

- Therefore $E(u,v)$ is a first integral for (8).

$\boxed{\text{Recalling the global well-posedness}}$

• In view of the Hamiltonian structure and the properties of the linear equation, a natural phase space for

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3. \tag{9}$$

is

$$\mathcal{H}^s(\mathbb{T}^3) \equiv H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)\,.$$

Theorem 8 (the classical result again)
Let $s \geq 1$. For every $(u_0, v_0) \in \mathcal{H}^s(\mathbb{T}^3)$ there is a unique solution of (9) in $C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^3))$.

• Denote by $\Phi(t) : \mathcal{H}^s(\mathbb{T}^3) \to \mathcal{H}^s(\mathbb{T}^3)$ the resulting flow in Theorem 8.

• **We are interested in the statistical description of $\Phi(t)$.**

Theorem 9 (Gunaratnam-Oh-Tz.-Weber (2019))
Let $\sigma \geq 4$ be an even integer. Then μ_σ is quasi-invariant under the nonlinear flow $\Phi(t)$.

- This result was perviously obtained by Tz. for $\mathbb{T}^1$ and Oh-Tz. for $\mathbb{T}^2$.
- For $d = 1, 2$ we prove more than for $d = 3$.
- For $d = 2, 3$ renormalisation arguments are needed.
- The result of Theorem 9 holds also for the cubic wave equation

$$\partial_t^2 u - \Delta u + u^3 = 0\,.$$

A corollary

Theorem 10 (a stability property)
Let $\sigma \geq 4$ be an even integer. Let

$$f_1(u,v), f_2(u,v) \in L^1(d\mu_\sigma(u,v))$$

and let $\Phi(t)$ be the flow of

$$\partial_t u = v, \quad \partial_t v = \Delta u - u - u^3,$$

defined μ_σ a.s. Then for every $t \in \mathbb{R}$, the transports of the measures

$$f_1(u,v) d\mu_\sigma(u,v), \quad f_2(u,v) d\mu_\sigma(u,v)$$

by $\Phi(t)$ are given by

$$F_1(t,u,v) d\mu_\sigma(u,v), \quad F_2(t,u,v) d\mu_\sigma(u,v)$$

respectively, for suitable $F_1(t,\cdot), F_2(t,\cdot) \in L^1(d\mu_\sigma)$. Moreover

$$\|F_1(t) - F_2(t)\|_{L^1(d\mu_\sigma)} = \|f_1 - f_2\|_{L^1(d\mu_\sigma)} .$$

Further comments

• The proof crucially exploits the "dispersion" for any σ. More precisely, μ_σ are not quasi-invariant under the flow of

$$\partial_t u = v, \quad \partial_t v = -u - u^3,$$

as shown in a recent work by Sosoe-Trenberth-Xiao.

• I would love to be able to prove that the same result should hold for any $\sigma > 1/2$ (for $\sigma \in (1/2, 1]$ one should use a probabilistic global well-posedness in the sense of Burq-Tz.).

$\boxed{\text{More about the gaussian measures } \mu_\sigma}$

• μ_σ is **formally** defined by

$$d\mu_\sigma = Z_\sigma^{-1} e^{-\frac{1}{2}\|(u,v)\|^2_{\mathcal{H}^{\sigma+1}}} du dv$$

or

$$Z_\sigma^{-1} \prod_{n\in\mathbb{Z}^3} e^{-\frac{1}{2}\langle n\rangle^{2(\sigma+1)}|\widehat{u}_n|^2} e^{-\frac{1}{2}\langle n\rangle^{2\sigma}|\widehat{v}_n|^2} d\widehat{u}_n d\widehat{v}_n \,,$$

where $\widehat{u}_n$ and $\widehat{v}_n$ denote the Fourier transforms of u and v respectively.

• For $\sigma > 3/2$, it is a gaussian measure on $L^2(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ with covariance operator

$$((1-\Delta)^{-\sigma-1}, (1-\Delta)^{-\sigma}).$$

More about the gaussian measures μ_σ (sequel)

• Our way to define μ_σ rigorously is to see it as the induced probability measure under the map

$$\omega \longmapsto (u^\omega(x), v^\omega(x))$$

with

$$u^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}, \qquad v^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x}\ . \tag{10}$$

• The partial sums of the series in (10) are a Cauchy sequence in $L^2(\Omega; \mathcal{H}^s(\mathbb{T}^3))$ for every $s < \sigma + 1 - \frac{3}{2}$ and therefore one can see μ_σ as a probability measure on $\mathcal{H}^s$ for a fixed $s < \sigma + 1 - \frac{3}{2} = \sigma - \frac{1}{2}$.

• For the same range of s, the triplet $\Big(\mathcal{H}^{\sigma+1}(\mathbb{T}^3), \mathcal{H}^s(\mathbb{T}^3), \mu_\sigma\Big)$ forms an abstract Wiener space.

$\boxed{\text{Related results 1 (Cameron-Martin 1944)}}$

Theorem 11 (Cameron-Martin in the context of the measure μ_σ) *For a fixed $(h_1, h_2) \in \mathcal{H}^s$, $s < \sigma - \frac{1}{2}$, the transport of μ_σ under the shift*

$$(u_1, u_2) \longmapsto (u_1, u_2) + (h_1, h_2)$$

is absolutely continuous with respect to μ_σ if and only if

$$(h_1, h_2) \in \mathcal{H}^{\sigma+1}.$$

$\boxed{\text{Our result in the context of Cameron-Martin's theorem}}$

• For $(u, v) \in \mathcal{H}^s$, we classically have

$$\Phi(t)(u, v) = \bar{S}(t)\Big((u, v) + (h_1, h_2)\Big),$$

where $(h_1, h_2) = (h_1(u, v), h_2(u, v)) \in \mathcal{H}^{s+1}$ (one smoothing and not more).

• Clearly if $s < \sigma - \frac{1}{2}$ then $s + 1 < \sigma + 1$ and therefore our result displays a remarkable property of the vector field generating $\Phi(t)$.

• More precisely, if (h_1, h_2) were independent of (u, v) of regularity $\mathcal{H}^{s+1}$ then the transported measure would not be absolutely continuous with respect to μ_σ !

Related results 2 (Ramer 1974)

• For $s < \sigma - \frac{1}{2}$, let us consider a diffeo Φ on $\mathcal{H}^s(\mathbb{T}^3)$ of the form

$$\Phi(u,v) = (u,v) + F(u,v),$$

where $F : \mathcal{H}^s(\mathbb{T}^3) \to \mathcal{H}^{\sigma+1}(\mathbb{T}^3)$. Suppose that

$$DF(u,v) : \mathcal{H}^{\sigma+1}(\mathbb{T}^3) \to \mathcal{H}^{\sigma+1}(\mathbb{T}^3)$$

is Hilbert-Schmidt.

• Ramer (1974) : under the above assumption μ_σ is quasi-invariant under Φ.

• Typical example :

$$F(u,v) = \varepsilon(1-\Delta)^{-3/2-\delta}(u^2, v^2), \quad \delta > 0,\ |\varepsilon| \ll 1,$$

i.e. 3-smoothing is needed.

• The Ramer's result would apply in the context of

$$\partial_t^2 u + (-\Delta)^\alpha u + u + u^3 = 0, \quad \alpha > 3.$$

• Therefore our result *seems* to go much beyond Ramer's framework.

$\boxed{\text{Related results 3. (Cruzeiro 1983)}}$

• In her work Ana Bela Cruzeiro considers a general equation of the form

$$\partial_t u = X(u),$$

where X is a vector field on $\mathcal{H}^s$, $s < \sigma - \frac{1}{2}$.

• A.B. Cruzeiro 1983 : the resulting flow has μ_σ as a quasi-invariant measure provided that several assumptions are satisfied, the most important being

$$\int_{\mathcal{H}^s} e^{\mathrm{div}(X(u))} d\mu_\sigma(u) < \infty. \tag{11}$$

• Very very roughly speaking, our work consists in verifying in practice a conditions of type (11) by exploiting techniques from dispersive PDE's.

A connection with the wave turbulence type problems

• In the WT problems one studies the behaviour of

$$N(n,t)=\mathbb{E}\Big(|\mathcal{F}(\Phi(t)(u_0^\omega,v_0^\omega))(n)|^2\Big),\qquad n\in\mathbb{Z}^3,\,t\in\mathbb{R}$$

in various limits.

• Our results says that there exists a density (resulting from the quasi-invariance)

$$F(t,\omega)\geq 0,\qquad F(t,\cdot)\in L^1(\Omega),\qquad F(0,\omega)=1$$

such that

$$N(n,t)=\int_\Omega\Big(\frac{|g_n(\omega)|^2}{\langle n\rangle^{2\sigma+2}}+\frac{|h_n(\omega)|^2}{\langle n\rangle^{2\sigma}}\Big)F(t,\omega)dp(\omega)\,.$$

• Therefore the density $F(t,\omega)$ (if it exists !) contains all the information needed to know $N(n,t)$. It contains even more information.

A connection with the wave turbulence type problems (sequel)

- Therefore the density $F(t, \omega)$ (if it exists !) contains all the information needed to know $N(n, t)$. It contains even more information.
- Recent work by Debussche-Tsutsumi, Genovese-Luca-Tz. (in progress) and Planchon-Visciglia-Tz. (in progress) allow to know some precise informations on the densities $F(t, \omega)$.
- Therefore it does not seem impossible to me to study the WT limits directly in the densities of the quasi-invariance results. It looks to be an interesting line of research.

A dual formulation

• Denote by $u(t,x,\omega)$ the solution of the nonlinear wave equation with data

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{\sigma+1}} e^{in\cdot x}, \qquad v_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\sigma}} e^{in\cdot x} \ .$$

• Then there exists a density

$$f(t,\omega)\geq 0, \quad f(t,\cdot)\in L^1(\Omega), \quad f(0,\omega)=1$$

such that:

$$\int_\Omega |\widehat{u}_n(t,\omega)|^2 f(t,\omega)dp(\omega) = \langle n\rangle^{-2(\sigma+1)}\,, \quad \forall\, n\in\mathbb{Z}^3$$

where $u_n(t,\omega)$ are the Fourier coefficients of $u(t,x,\omega)$.

• I do not know whether this remark may be of some interest in wave turbulence considerations.

Corresponding results for NLS

• In a work by Planchon-Visciglia-Tz. the previous quasi-invariance results are extended to the $1d$ NLS. The NLS is harder than the wave equation because of the lack of direct smoothing. However, we can exploit some hidden smoothing via modified energies ...

• The extension to $2d$ is a challenging problem. Namely can we prove that the measure induced by the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x} \tag{12}$$

is quasi-invariant under the flow of the $2d$ NLS

$$i\partial_t u + \Delta u = |u|^2 u \text{ ?}$$

(which is perfectly well-defined for data given by (12) as far as $s > 2$).

• I would be happy to know the answer of this question even only for very large values of s.

Thank you very much !

(hoping that better times will come soon)