7ECM, Berlin,
July 22, 2016

Transverse stability issues in Hamiltonian PDE

Nikolay Tzvetkov

(University of Cergy-Pontoise)

The KdV equation

• The Korteweg- de Vries (KdV) equation was derived at the end of the 19'th century as an asymptotic model from the much more complicated (but derived from first principles !) water-waves system. The KdV equation reads

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0,$$

where the unknown u is a real valued function.

• The KdV solitary waves are the following particular solutions

$$S_c(t,x) = cQ(\sqrt{c}(x-ct)), \quad c>0, \quad Q(x) = 3\mathrm{ch}^{-2}(x/2)\,.$$

• $c\,Q(\sqrt{c}\,x)$ is a stationary solution of

$$\partial_t u - c\partial_x u + u\partial_x u + \partial_x^3 u = 0.$$

• What about the stability of $S_c(t,x)$?

Stability of the KdV solitary waves

Theorem 1 (Benjamin 1972)
The solitary wave $S_c(t,x)$ is (orbitally) stable as a solution of the KdV equation. More precisely, for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $u_0 \in H^1(\mathbb{R})$ such that

$$\|u_0(x) - S_c(0,x)\|_{H^1} < \delta$$

the solution of the KdV equation with initial datum u_0 satisfies

$$\inf_{a\in\mathbb{R}} \|u(t,x-a) - S_c(t,x)\|_{H^1} < \varepsilon, \quad \forall\, t \in \mathbb{R}.$$

• The Sobolev spaces $H^1(\mathbb{R}) \equiv \{u \in L^2(\mathbb{R}) \,:\, u' \in L^2(\mathbb{R})\}$ measuring the stability phenomenon is naturally imposed by the conservation laws of KdV.

• The global well-posedness of KdV in $H^1(\mathbb{R})$ is due to Kenig-Ponce-Vega.

• Asymptotic stability : Pego-Weinstein (1992), Martel-Merle (2001).

The KP models

• When studying the stability of the KdV solitary waves under transverse perturbations, the Soviet physicists Kadomtsev and Petviashvily introduced in 1970 the two dimensional models

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u) \pm \partial_y^2 u = 0$$

called KP-I and KP-II equations depending on the sign in front of $\partial_y^2 u$.

• The sign plus gives KP-II while the sign minus gives KP-I.

• $S_c(t,x)$ is a solution of the KP equations. The *remarkable formal analysis* of KP led to the conjecture that the KdV solitary wave was stable as a solution of the KP-II equation and that it was unstable as a solution of the KP-I equation.

• A mathematically rigorous proof of such statements was out of reach in 1970, in particular because of the lack of an analytic framework for defining the KP dynamics (at least close to the solitary waves).

The KP models (sequel)

• The natural idea we adopted in our works for an analytic framework in the studying of the KP equations was to consider these equations posed on the the product space $\mathbb{R}\times\mathbb{T}$, i.e. for $x\in\mathbb{R}$ and $y\in\mathbb{T}$, where $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$ denotes a one dimensional torus.

• In other words, we consider solutions of the KP equations which are localised in x (as $S_c(t,x)$ is) and periodic in the transverse variable y with period 2π.

• The choice of 2π is not canonical and any other period can be considered as well.

• However, if u is a solution of the KP equations then so is

$$u_\lambda(t,x,y)=\lambda^2 u(\lambda^3 t,\lambda x,\lambda^2 y),\quad \forall\,\lambda>0.$$

Thus we can always reduce the matters to the period 2π.

Results for KP-I

• The L^2 norm is (at least formally) conserved by the flow of the KP-I equation. So is the energy

$$E(u) = \int_{\mathbb{R}\times\mathbb{T}} (\partial_x u)^2 + \int_{\mathbb{R}\times\mathbb{T}} (\partial_x^{-1}\partial_y u)^2 - \frac{1}{3}\int_{\mathbb{R}\times\mathbb{T}} u^3 .$$

• Inspired by the structure of the KP-I conservation laws, we can define the spaces $Z^s = Z^s(\mathbb{R}\times\mathbb{T})$ as

$$Z^s = \{u \,:\, \|(1+|\xi|^s+|\xi^{-1}k|^s)\widehat{u}(\xi,k)\|_{L^2(\mathbb{R}_\xi\times\mathbb{Z}_k)} < \infty\}$$

and equipped with the natural norm (here by $\widehat{u}(\xi,k)$ we denote the Fourier transform of functions on the product space $\mathbb{R}\times\mathbb{T}$).

Theorem 2 (Ionescu-Kenig 2007)
The KP-I equation is globally well-posed in $Z^2(\mathbb{R}\times\mathbb{T})$.

• The proof is based on a method introduced by Koch-Tz. for studying low regularity well-posedness of dispersive PDE's with strong derivative losses in the non linear interactions.

Results for KP-I (sequel)

Theorem 3 (Zakharov 1975, Rousset-Tz. 2009)
The KdV solitary wave $S_c(t,x)$ is orbitally unstable as a solution of the KP-I equation, provided $c > 4/\sqrt{3}$. More precisely, for every $s \geq 0$ there exists $\eta > 0$ such that for every $\delta > 0$ there exists $u_0^\delta \in Z^2 \cap H^s$ and a time $T^\delta \approx |\log \delta|$ such that

$$\|u_0^\delta(x,y) - S_c(0,x)\|_{H^s(\mathbb{R}\times\mathbb{T})} + \|u_0^\delta(x,y) - S_c(0,x)\|_{Z^2(\mathbb{R}\times\mathbb{T})} < \delta$$

and the (global) solution of the KP-I equation satisfies

$$\inf_{a\in\mathbb{R}} \|u(T^\delta, x-a, y) - S_c(T^\delta, x)\|_{L^2(\mathbb{R}\times\mathbb{T})} > \eta.$$

- The proof by Rousset-Tz. uses a quite general method for constructing approximate solutions close to the solitary waves due to Grenier. One needs a soft energy estimate and location of the unstable modes analysis which in turn implies the crucial semi-group estimates.
- The Zakharov proof is different and seems to only work for some particular values of c. It is based on constructing an explicit solution of the KP-I equation (related to the integrability features of KP-I).

Results for KP-I (sequel)

Theorem 4 (Rousset-Tz. 2012)
The KdV solitary wave $S_c(t,x)$ is orbitally stable as a solution of the KP-I equation, provided $c < 4/\sqrt{3}$. More precisely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial datum u_0 of the KP-I equation satisfies $u_0 \in Z^2(\mathbb{R} \times \mathbb{T})$ and

$$\|u_0(x,y) - S_c(0,x)\|_{Z^1(\mathbb{R}\times\mathbb{T})} < \delta$$

then the solution of the KP-I equation with datum u_0 satisfies

$$\sup_{t\in\mathbb{R}} \inf_{a\in\mathbb{R}} \|u(t,x-a,y) - S_c(t,x)\|_{Z^1(\mathbb{R}\times\mathbb{T})} < \varepsilon.$$

- The study of the critical speed ($c = 4/\sqrt{3}$) solitary waves is a delicate open problem.

Results for KP-II

• The L^2 norm is (at least formally) conserved by the flow of the KP-II equation. So is the energy

$$E(u) = \int_{\mathbb{R}\times\mathbb{T}} (\partial_x u)^2 - \int_{\mathbb{R}\times\mathbb{T}} (\partial_x^{-1}\partial_y u)^2 - \frac{1}{3}\int_{\mathbb{R}\times\mathbb{T}} u^3 .$$

• Therefore the L^2 norm is the only useful a priori bound for KP-II. This makes the analysis quite involved.

Theorem 5 (Molinet-Saut-Tz. 2011)
The KP-II equation

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u) + \partial_y^2 u = 0, \qquad x \in \mathbb{R},\ y \in \mathbb{T}$$

is globally well-posed in $L^2(\mathbb{R}\times\mathbb{T})$.

• The proof is based on a delicate use of the Bourgain $X^{s,b}$ spaces.

Results for KP-II (sequel)

Theorem 6 (Mizumachi-Tz. 2012)
The KdV solitary wave $S_c(t,x)$ is orbitally stable as a solution of the KP-II equation for all $c>0$: for every $\varepsilon>0$, there exists $\delta>0$ such that if the initial datum u_0 of the KP-II equation satisfies

$$\|u_0(x,y)-S_c(0,x)\|_{L^2(\mathbb{R}\times\mathbb{T})}<\delta$$

then the solution of the KP-II equation with datum u_0 satisfies

$$\sup_{t\in\mathbb{R}}\inf_{a\in\mathbb{R}}\|u(t,x-a,y)-S_c(t,x)\|_{L^2(\mathbb{R}\times\mathbb{T})}<\varepsilon.$$

Moreover, there is also an asymptotic stability in the following sense. There exists a constant $\tilde{c}$ satisfying $\tilde{c}-c=O(\delta)$ and a modulation parameter $x(t)$ satisfying $\lim_{t\to\infty}\dot{x}(t)=\tilde{c}$ and such that

$$\lim_{t\to\infty}\|u(t,x,y)-S_{\tilde{c}}(0,x-x(t))\|_{L^2((x\geq ct/10)\times\mathbb{T}_y)}=0.$$

• For $u_0(x,y)$ independent of y, we recover a result of Merle-Vega for the L^2 stability of the KdV solitary waves.

Extensions to the water-waves system

When we study solitary waves of speed c, we face the problem

$$\partial_t \eta = \partial_x \eta + G[\eta]\varphi,$$
$$\partial_t \varphi = \partial_x \varphi - \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\frac{(G[\eta]\varphi + \nabla\varphi\cdot\nabla\eta)^2}{1+|\nabla\eta|^2} - \alpha\eta + \beta\nabla\cdot\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}},$$

where $\eta = \eta(t,x,y)$, $\varphi = \varphi(t,x,y)$, $t,x,y \in \mathbb{R}$, $\nabla = (\partial_x, \partial_y)$ and

$$\alpha = \frac{gh}{c^2}, \quad \beta = \frac{b}{hc^2}.$$

g is the gravity constant, b takes into account the surface tension effects, h represents the deepness of the fluid domain and $G[\eta]$ is a Dirichlet-Neumann map. $G[\eta]$ is a first order pseudo-differential operator with principal symbol

$$\Big((1+|\nabla\eta|^2)|\xi|^2 - (\nabla\eta\cdot\xi)^2\Big)^{\frac{1}{2}}, \ \xi \in \mathbb{R}^2.$$

Existence of solitary waves for the water-waves system

Theorem 7 (Amick-Kirchgässner 1989)
Suppose that $\alpha = 1 + \varepsilon^2$ *and* $\beta > 1/3$*. Then there exists* ε_0 *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *there is a stationary solution* $(\eta_\varepsilon(x), \varphi_\varepsilon(x))$ *of the water-waves problem of the form*

$$\eta_\varepsilon(x) = \varepsilon^2 \Theta(\varepsilon x, \varepsilon), \quad \varphi_\varepsilon(x) = \varepsilon \Phi(\varepsilon x, \varepsilon)\,.$$

We have that η_ε *is exponentially localised and*

$$\Theta(y, 0) = -\mathrm{ch}^{-2}\Big(\frac{y}{2(\beta - 1/3)^{1/2}}\Big)\,.$$

• Observe that the solitary waves established by the above result are of speed essentially $\sqrt{gh}$.
• Mielke (2002) proved the analogue of the Benjamin result concerning the stability of these solitary waves (under un assumptions of global well-posedness close to the solitary waves).

Transverse instability for the water-waves system

Theorem 8 (Rousset-Tz. 2011)
Suppose that $\alpha = 1 + \varepsilon^2$ *and* $\beta > 1/3$. *There exists* ε_0 *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *there is* $L_0 > 0$ *such that for* $L > L_0$ *the following holds true. For every* $s \geq 0$, *there exists* $\kappa > 0$ *such that for every* $\delta > 0$, *there exist* $(\eta_0^\delta(x,y), \varphi_0^\delta(x,y))$ *and a time* $T^\delta \sim |\log \delta|$ *such that*

$$\|(\eta_0^\delta(x,y), \varphi_0^\delta(x,y)) - (\eta_\varepsilon(x), \varphi_\varepsilon(x))\|_{H^s(\mathbb{R}\times\mathbb{T}_L)\times H^s(\mathbb{R}\times\mathbb{T}_L)} \leq \delta$$

and a solution $(\eta^\delta(t,x,y), \varphi^\delta(t,x,y))$ *of the water-waves system, posed on* $\mathbb{R}\times\mathbb{T}_L$ *with initial datum* $(\eta_0^\delta, \varphi_0^\delta)$, *defined on* $[0, T^\delta]$ *and satisfying*

$$\inf_{a\in\mathbb{R}} \|(\eta^\delta(T^\delta,x,y), \varphi^\delta(T^\delta,x,y)) - (\eta_\varepsilon(x-a), \varphi_\varepsilon(x-a))\|_{L^2(\mathbb{R}\times\mathbb{T}_L)\times L^2(\mathbb{R}\times\mathbb{T}_L)} > \kappa.$$

Open problems

- The critical speed problem for KP-I.
- We believe that KP-I is well-posed in Z^1. This would relax the assumption on the perturbation in the stability statement for sub-critical speeds.
- Asymptotic stability for KP-I for for sub-critical speeds.
- We believe that there is a conditional small period stability for the water-waves system (in the spirit of the work by Mielke).
- We also believe that we can have an unconditional statement in Mielke's analysis for finite but long time scales, depending on the size of the initial perturbation.
- Stability results for the water-waves system in the KP-II regime. For instance, one may try to extend the quite flexible approachof Pego-Weinstein to the water-waves system.

Thank you for the attention !