Concerning the pathological set in the context of probabilistic well-posedness

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(based joint work with Chenmin Sun)
The nonlinear wave equation

- In this talk, we consider the Cauchy problem for the nonlinear wave equation, posed on the 3d torus $\mathbb{T}^3$:
  \[(\partial_t^2 - \Delta)u + u^3 = 0, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x). \quad (1)\]
- The energy
  \[\int_{\mathbb{T}^3} \left((\partial_t u)^2 + |\nabla u|^2\right) + \frac{1}{2} \int_{\mathbb{T}^3} u^4\]
is formally conserved by (1). For $s \in \mathbb{R}$, we set
  \[\mathcal{H}^s(\mathbb{T}^3) := H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)\]
which is a natural phase space for (1).

**Theorem 1 (classical)**
- For every $(u_0, u_1) \in \mathcal{H}^1(\mathbb{T}^3)$ there exists a unique global solution of (1) in the class $(u, \partial_t u) \in C(\mathbb{R}; \mathcal{H}^1(\mathbb{T}^3))$.
- If in addition $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$ for some $s \geq 1$ then
  \[(u, \partial_t u) \in C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^3)).\]
- The dependence with respect to the initial data is continuous.
Lower regularity using the dispersion

- Thanks to the Strichartz estimates the local in time part of Theorem 1 can be extended to the class $\{(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3), s \geq 1/2\}$.
- The global in time part can be extended to $s > 3/4$ using almost conservation law techniques:

**Theorem 2 (Kenig-Ponce-Vega, Gallagher-Planchon, Roy)**

Let $s > 3/4$ and fix $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$. Let $(u_{0,n}, u_{1,n})_{n \in \mathbb{N}}$ be any sequence of smooth data approximating $(u_0, u_1)$ in $\mathcal{H}^s(\mathbb{T}^3)$ and let $u_n(t, x)$ be the smooth solution of

$$(\partial_t^2 - \Delta) u_n + u_n^3 = 0, \quad u|_{t=0} = u_{0,n}, \quad t u|_{t=0} = u_{1,n}.$$  

Then there exists a limit object $u(t)$ such that for any $T > 0$,

$$\lim_{n \to \infty} \| (u_n(t), \partial_t u_n(t)) - (u(t), \partial_t u(t)) \|_{L^\infty([-T,T];\mathcal{H}^s(\mathbb{T}^3))} = 0.$$  

Moreover $u(t)$ solves the nonlinear wave equation in the sense of distributions.

- We conjecture that Theorem 2 remains true for $s \geq 1/2$ (proved recently by Dodson in the radial case of $\mathbb{R}^3$).
Limit of the deterministic methods

Theorem 3
Let $s \in (0, 1/2)$ et $(u_0, u_1) \in \mathcal{H}^s(T^3)$. There exists a sequence

$$u_N(t, x) \in C^\infty(\mathbb{R} \times T^3), \quad N = 1, 2, \ldots$$

such that

$$(\partial_t^2 - \Delta)u_N + u_N^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N(0) - u_0, \partial_t u_N(0) - u_1)\|_{\mathcal{H}^s(T^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \|u_N(t)\|_{L^\infty([-T,T];H^s(T^3))} = +\infty.$$  

• The proof is based on an idea introduced by Gilles Lebeau and further developed by Christ-Colliander-Tao, Burq-Gérard–Tz., Xia.
Solving the equation by probabilistic methods

• We can ask whether some form of well-posedness survives for initial data in $\mathcal{H}^s(\mathbb{T}^3)$, $s < 1/2$?

• The answer of this question is positive if we endow the space $\mathcal{H}^s(\mathbb{T}^3)$, $s < 1/2$ with a non degenerate probability measure such that we have the existence, the uniqueness, and a form of continuous dependence almost surely with respect to this measure.
Probabilistic global well-posedness

**Theorem 4 (Burq-Tz. 2008)**

Let $0 \leq s < \frac{1}{2}$. Then there is a dense set $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ satisfying $\Sigma \cap \mathcal{H}^{s'}(\mathbb{T}^3) = \emptyset$ for every $s' > s$ such that the following holds true. For every $(f, g) \in \Sigma$, denote by $(u_n(t))_{n \in \mathbb{N}}$ the smooth solutions of

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u(0, x) = \rho_n * f, \quad \partial_t u(0, x) = \rho_n * g,$$

where $(\rho_n)_{n \in \mathbb{N}}$ is an approximate identity. Then there exists a limit object $u(t)$ such that for any $T > 0$,

$$\lim_{n \to \infty} \| (u_n(t), \partial_t u_n(t)) - (u(t), \partial_t u(t)) \|_{L^\infty([-T, T]; \mathcal{H}^s(\mathbb{T}^3))} = 0.$$ 

Moreover $u(t)$ solves the nonlinear wave equation in the distributional sense.
Comments

• The proof of the previous result is inspired by the seminal contribution of 1994 by Bourgain. There are however several new features:

• The first one is that more general randomisations are allowed. This led to similar results in the context of a non compact spatial domains.

• The argument allowing to pass from local to global solutions is based on a probabilistic energy estimates (Gronwall method) while the argument giving the globalisation of the local solutions in the Bourgain work is restricted to a very particular distribution of the initial data related to the Gibbs measure (measure preserving method).

• The result by Burq-Tz. deals with functions of positive Sobolev regularity which avoids a renormalization of the equation, making the results more natural from a purely PDE perspective. In particular, it is clear what ill-posedness means.
Comments (sequel)

- It would very interesting to prove such type of results for quasi-linear wave equations.
- A first result in this direction was obtained by B. Bringmann who proved probabilistic well-posedness for equations of type

\[ (\partial_t^2 - \Delta)u = F(\nabla u), \]

and a quadratic $F$. 
The pathological set

• The result by Burz-Tz. provides a nice dense set $\Sigma$ of initial data such that for good approximations we get nice global solutions (but for bad approximations we get divergent sequences!).
• What about initial data outside $\Sigma$?

**Theorem 5 (Sun-Tz. 2020)**

Let $0 < s < \frac{1}{2}$. Then there is an approximate identity $(\rho_n)_{n \in \mathbb{N}}$ and there is a dense set $S \subset \mathcal{H}^s(\mathbb{T}^3)$ such that for every $(f, g) \in S$, the family of the smooth solutions of

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u(0, x) = \rho_n * f, \quad \partial_t u(0, x) = \rho_n * g$$

do not converge. More precisely, for every $T > 0$,

$$\lim_{n \to \infty} \|u_n(t)\|_{L^\infty([0,T];\mathcal{H}^s(\mathbb{T}^3))} = +\infty.$$
Consider again

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u(0, x) = \rho_n * f, \quad \partial_t u(0, x) = \rho_n * g \quad (2)$$

and let $\mathcal{P}$ be the set of $(f, g) \in \mathcal{H}^s(\mathbb{T}^3)$ such that the solution $u_n$ of (2) satisfies the property

$$\limsup_{n \to \infty} \|u_n(t)\|_{L^\infty([0,1];\mathcal{H}^s(\mathbb{T}^3))} = +\infty.$$

**Corollary 6**

*The set $\mathcal{P}$ contains a dense $G_\delta$ subset of $\mathcal{H}^s(\mathbb{T}^3)$.*

Consequently, by the Baire category theorem, the good data set $\Sigma$ in the Burq-Tz. theorem is not a $G_\delta$ subset of $\mathcal{H}^s(\mathbb{T}^3)$.

On the other hand, the pathological sets are negligible with respect to the measures introduced by the natural gaussian fields used in the probabilistic well-posedness results.
• The previous discussion confirms that the topological and the measure theoretic notions of genericity are very different.

• For examples of $G_\delta$ dense sets giving solutions of Hamiltonian PDE’s with growing Sobolev norms for large times, we refer to the works by Hani and Grellier-Gérard.

• In our result, the Sobolev norms are growing in very short times, depending on the frequency localization of the initial data.

• Naive question: Is such a phenomenon present in the context of the Lindblad ill-posedness results for quasi-linear wave equations?
The ODE profile

• The basic idea is that since the regularity is supercritical the linear part of the equation is treated as a perturbation.
• Therefore, we consider the solution of the ODE

\[ V'' + V^3 = 0, \quad V(0) = 1, \quad V'(0) = 0 \]

which is globally defined and periodic (oscillating between 0 and 1).
• Let us fix the the positive bump functions \( \rho, \varphi \in C^\infty(\mathbb{R}^3) \), supported in \( |x| \leq \frac{1}{100} \), seen as functions on \( \mathbb{T}^3 \) with \( \int \rho = 1 \). As usual, \( \rho_\epsilon(x) := \epsilon^{-3} \rho(x/\epsilon) \). Let

\[ v_n(0, x) := \kappa_n n^{\frac{3}{2}-s} \varphi(nx), \quad v_\epsilon^e(0, x) := \rho_\epsilon * v_n(0, x), \quad \kappa_n = (\log n)^{-\delta_1}, \quad \delta_1 > 0. \]

• Define

\[ v_\epsilon^e(t, x) = v_\epsilon^e(0, x)V(t v_\epsilon^e(0, x)). \]

Then one verifies that \( v_\epsilon^e \) solves the dispersionless equation

\[ \partial_t^2 v_\epsilon^e + (v_\epsilon^e)^3 = 0, \quad (v_\epsilon^e, \partial_t v_\epsilon^e)|_{t=0} = (v_\epsilon^e(0, x), 0). \]
Proposition 7

Let $0 < s < \frac{1}{2}$. Set

$$\epsilon_n = \frac{1}{100n}, \quad t_n = (\log n)^{\delta_2} n^{-(\frac{3}{2} - s)}, \quad \delta_2 > 0.$$ 

Then we have the lower bound

$$\|v_n^{\epsilon_n}(t_n)\|_{H^s(T^3)} \gtrsim \kappa_n (\log n)^{\delta_2 - \delta_1} n^{s}$$

and the upper bounds

$$\|v_n^{\epsilon_n}(t)\|_{H^k(T^3)} \lesssim \kappa_n (\log n)^{\delta_2 - \delta_1} n^{k} n^{s}, \quad k = 0, 1, 2, 3, \cdots, \quad t \in [0, t_n],$$

$$\|\partial^\alpha v_n^{\epsilon_n}(t)\|_{L^\infty(T^3)} \lesssim (\log n)^{\delta_2 - \delta_1} n^{s} \kappa_n n^{\frac{3}{2} - s}, \quad \alpha \in \mathbb{N}^3, |\alpha| = 0, 1, \quad t \in [0, t_n].$$
The perturbative analysis

- For \((u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)\), denote by \(u_n^{\epsilon_n}\) the solution of
  \[
  \partial_t^2 u_n^{\epsilon_n} - \Delta u_n^{\epsilon_n} + (u_n^{\epsilon_n})^3 = 0
  \]
with the initial data
  \[
  (u_n^{\epsilon_n}(0), \partial_t u_n^{\epsilon_n}(0)) = \rho_{\epsilon_n} \ast \left( (u_0, u_1) + (v_n(0), 0) \right).
  \]

**Proposition 8**

Assume that \(0 < s < \frac{1}{2}\). Then for any \(0 < \theta < \frac{1}{2}(\frac{1}{2} - s)\) there exist \(C > 0, \delta_2 > 0\), such that for any \(\delta_1 \in (0, \delta_2)\), we have

\[
\sup_{t \in [0, t_n]} \|u_n^{\epsilon_n}(t) - \rho_{\epsilon_n} \ast S(t)(u_0, u_1) - v_n^{\epsilon_n}(t)\|_{H^\nu(\mathbb{T}^3)} \leq C n^{(\nu-s)-\theta}, \quad \forall \nu = 0, 1, 2,
\]

where \(S(t)\) is the free evolution and the constant \(C\) only depends on the smooth data \((u_0, u_1)\) and \(\theta > 0\). Consequently, we have

\[
\sup_{t \in [0, t_n]} \|u_n^{\epsilon_n}(t) - \rho_{\epsilon_n} \ast S(t)(u_0, u_1) - v_n^{\epsilon_n}(t)\|_{H^s(\mathbb{T}^3)} \leq C n^{-\theta}.
\]

In particular, for \(\delta_1\) sufficiently small,

\[
\|u_n^{\epsilon_n}(t_n)\|_{H^s(\mathbb{T}^3)} \gtrsim (\log n)^{s(\delta_2 - \delta_1) - \delta_1} \to \infty, \text{ as } n \to \infty.
\]
Finite propagation speed of the nonlinear wave equation

Proposition 9
Let \( w_1, w_2 \) be two \( C^\infty \) solutions of the nonlinear wave equation
\[
\partial_t^2 w - \Delta w + w^3 = 0.
\]
If the initial data \((w_1(0), \partial_t w_1(0)), (w_2(0), \partial_t w_2(0))\) coincide on the ball
\[
B(x_0, r_0) := \{ x \in \mathbb{R}^3 : |x - x_0| \leq r_0 \}
\]
then for \( 0 \leq t < r_0 \),
\[
(w_1(t), \partial_t w_1(t)) = (w_2(t), \partial_t w_2(t))
\]
on \( B(x_0, r_0 - t) \).
Proof of the main result

• We identify $\mathbb{T}^3$ with $[-\pi, \pi]^3$ and we use the coordinate system $x = (x_1, x')$ near the origin. Let $z^k = (\frac{1}{k}, 0, 0)$
• Let $n_k = e^{e^k}$, and define

$$v_{0,k}(x) := (\log n_k)^{-\delta_1} n_k^{\frac{3}{2} - s} \varphi(n_k(x_1 - \frac{1}{k}), n_k x') = v_{n_k}(0, \cdot - z^k).$$

There exists $k_0$, such that for all $k \geq k_0$, the supports of $v_{0,k}$ are pairwise disjoint and for $k_0 \leq k_1 < k_2$,

$$\text{dist}(\text{supp}(v_{0,k_1}), \text{supp}(v_{0,k_2})) \sim \frac{1}{k_1} - \frac{1}{k_2}.$$

• Denote by $B_k = B(z^k, r_k)$, where $r_k = \frac{1}{k^3}$. For $k_0 \gg 1$, the balls $B_k, k \geq k_0$ are mutually disjoint. Moreover,

$$\text{supp}(\rho_{\epsilon_{n_k}} * v_{0,k}) \subset \tilde{B}_k,$$

where $\tilde{B}_k = B(z^k, \frac{r_k}{3})$ (recall that $\epsilon_{n_k} = \frac{1}{100n_k}$).
Proof of the main result (sequel)

• We have that

$$\text{dist}(\text{supp}(\rho_{\epsilon n_k} \ast (v_0 - v_{0,k})), B_k) \gtrsim \frac{1}{k^2},$$

where

$$v_0 = \sum_{k \geq k_0} v_{0,k} \in H^s(\mathbb{T}^3).$$

• In particular, for any \((u_0, u_1) \in C^\infty \times C^\infty\),

$$\rho_{\epsilon n_k} \ast ((u_0, u_1) + (v_0, 0))$$

coincides with

$$\rho_{\epsilon n_k} \ast ((u_0, u_1) + (v_{0,k}, 0))$$

on the ball \(B_k\).
Proof of the main result (definition of the pathological set)

• Let $k_0 \gg 1$, defined in the previous discussion. Set

$$S = C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3) + \left\{ \left( \sum_{k=k_1}^\infty v_{0,k}, 0 \right) : k_1 \geq k_0 \right\}.$$

• Since

$$\left\| \sum_{k=k_1}^\infty v_{0,k} \right\|_{H^s(\mathbb{T}^3)} \leq \sum_{k=k_1}^\infty \|v_{0,k}\|_{H^s(\mathbb{T}^3)} \leq \sum_{k=k_1}^\infty e^{-k\delta_1} \to 0 \text{ as } k_1 \to \infty,$$

we conclude that $S$ is dense in $\mathcal{H}^s(\mathbb{T}^3)$. 
Proof of the main result (sequel)

• Fix \((f, g) \in S\). By definition, there exists \((u_0, u_1) \in \mathcal{C}^\infty \times \mathcal{C}^\infty\) and \(k_1 \geq k_0\), such that

\[ (f, g) = (u_0, u_1) + \left( \sum_{k = k_1}^{\infty} v_{0,k}, 0 \right). \]

• Our goal is to show that, for any \(N > 0\) and any \(\delta > 0\), there exist \(\tau_N \in [0, T]\) and \(0 < \epsilon < \delta\), such that the solution \(u^\epsilon\) to our equation with initial data \(\rho^\epsilon \ast (f, g)\) satisfies

\[ \| u^\epsilon(\tau_N) \|_{H^s(T^3)} > N. \]  

(3)

• We will choose \(k \geq k_1\), large enough, such that

\[ \kappa n_k (\log n_k)^{(\delta_2 - \delta_1)s} \geq N, \quad \epsilon_k = \frac{1}{100n_k} < \delta. \]

This can be achieved by choosing \(\delta_1 < \delta_2\) such that \(s(\delta_2 - \delta_1) > \delta_1\).
Proof of the main result (sequel)

• Let \( \tilde{u}_k \) be the solution of our equation with the initial data
  \[ \rho \epsilon_n \ast (u_0, u_1) + \rho \epsilon_n \ast (v_0, 0). \]

• Let \( \tilde{v}_k \) be the solution of
  \[ \partial_t^2 \tilde{v}_k + (\tilde{v}_k)^3 = 0 \]
  with the initial data \( \rho \epsilon_n \ast (v_0, 0) \).

• We remark that \( \tilde{v}_k, \tilde{u}_k \) are just \( v_{\epsilon_n}, u_{\epsilon_n} \) the proposition of the perturbative analysis, up to translation.

• In particular,
  \[ \| \tilde{u}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)} \gtrsim (\log n_k)^{s(\delta_2 - \delta_1) - \delta_1}, \quad (4) \]
  and
  \[ \| \tilde{u}_k(t_{n_k}) - \rho \epsilon_n \ast S(t_{n_k})(u_0, u_1) - \tilde{v}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)} \lesssim n_k^{-\theta}. \quad (5) \]
Proof of the main result (sequel)

• We now apply the finite propagation speed property to $\tilde{u}_k$ and $u^{\epsilon_{nk}}$. At $t = 0$,

$$ (u^{\epsilon_{nk}}(0), \partial_t u^{\epsilon_{nk}}(0))|_{B_k} = (\tilde{u}_k(0), \partial_t \tilde{u}_k(0))|_{B_k}, $$

and therefore

$$ (u^{\epsilon_{nk}}(t), \partial_t u^{\epsilon_{nk}}(t))|_{B(z^k, r_k-t)} = (\tilde{u}_k(t), \partial_t \tilde{u}_k(t))|_{B(z^k, r_k-t)}, \quad \forall 0 \leq t < r_k. $$

• In particular, for large $k$,

$$ (u^{\epsilon_{nk}}(t), \partial_t u^{\epsilon_{nk}}(t))|_{B(z^k, r_k/2)} = (\tilde{u}_k(t), \partial_t \tilde{u}_k(t))|_{B(z^k, r_k/2)}, \quad \forall t \in [0, t_{nk}]. $$

• Take $\chi \in C_c^\infty(\mathbb{R}^3)$, such that $\chi(x) \equiv 1$ if $|x| < \frac{1}{3}$ and $\chi \equiv 0$ if $|x| \geq \frac{1}{2}$. Define $\chi_k(x) := \chi((x-z^k)/r_k)$, hence $\chi_k|_{\tilde{B}_k} \equiv 1$ and $\chi_k|_{B(z^k, r_k/2)^c} \equiv 0$. Therefore, we have

$$ \chi_k(x)(u^{\epsilon_{nk}}(t), \partial_t u^{\epsilon_{nk}}(t)) = \chi_k(x)(\tilde{u}_k(t), \partial_t \tilde{u}_k(t)), \quad \forall t \in [0, t_{nk}]. $$
Proof of the main result (sequel)

• Since $s < 1/2$, we can localise

$$\| u^{\epsilon n_k}(t_{n_k}) \|_{H^s(\mathbb{T}^3)} \gtrsim \| \chi_k u^{\epsilon n_k}(t_{n_k}) \|_{H^s(\mathbb{T}^3)} \sim \| \chi_k(x) \tilde{u}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)}. $$

• Next, we can write

$$\| \chi_k(x) \tilde{u}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)} \geq \| \tilde{u}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)} - \| (1 - \chi_k) \tilde{u}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)} $$

$$= \| \tilde{u}_k(t_{n_k}) \|_{H^s(\mathbb{T}^3)} - \| (1 - \chi_k)(\tilde{u}_k(t_{n_k}) - \tilde{v}_k(t_{n_k})) \|_{H^s(\mathbb{T}^3)},$$

where in the last equality, we crucially use the fact that

$$(1 - \chi_k)\tilde{v}_k(t_{n_k}) = 0,$$

thanks to the support property of $\tilde{v}_k$. 
Proof of the main result (sequel)

• Therefore, we have

\[ \|u^{\epsilon_{nk}}(t_{nk})\|_{H^s(\mathbb{T}^3)} \gtrsim \|\tilde{u}_k(t_{nk})\|_{H^s(\mathbb{T}^3)} - \|(1 - \chi_k)(\rho_{\epsilon_{nk}} \ast S(t_{nk})(u_0, u_1))\|_{H^s(\mathbb{T}^3)} \]

\[ - \|(1 - \chi_k)(\tilde{u}_k(t_{nk}) - \rho_{\epsilon_{nk}} \ast S(t_{nk})(u_0, u_1) - \tilde{v}_k(t_{nk}))\|_{H^s(\mathbb{T}^3)}. \]

• Consequently

\[ \|u^{\epsilon_{nk}}(t_{nk})\|_{H^s(\mathbb{T}^3)} \gtrsim (\log n_k)^{s(\delta_2 - \delta_1) - \delta_1 - C - n_k^{-\theta}}. \]

• It remains to choose \( \delta_1 > 0 \) small such that \( s(\delta_2 - \delta_1) - \delta_1 > 0 \) and \( k \gg 1 \).

• This completes the proof.
On the proof of the corollary

- Denote by $u^\varepsilon(t) = \Phi(t)(\rho \varepsilon \ast (f, g))$ the solution of the cubic wave equation with initial data $\rho \varepsilon \ast (f, g)$.
- The set

$$O := \{(f, g) \in H^s(\mathbb{T}^3) : \limsup_{k \to \infty} \|\Phi(t)(\rho \varepsilon_{nk} \ast (f, g))\|_{L^\infty([0,1]; H^s(\mathbb{T}^3))} = \infty\}$$

is contained in the pathological set $\mathcal{P}$.
- As a byproduct of the previous analysis, $S \subset O$, hence $O$ is dense.
- In addition

$$O = \bigcap_{N=1}^{\infty} O_N,$$

where

$$O_N := \{(f, g) \in H^s(\mathbb{T}^3) : \limsup_{k \to \infty} \|\Phi(t)(\rho \varepsilon_{nk} \ast (f, g))\|_{L^\infty([0,1]; H^s(\mathbb{T}^3))} > N\}.$$
On the proof of the corollary (sequel)

- By definition,

\[ O_N = \bigcap_{k_0=1}^{\infty} \bigcup_{k=k_0}^{\infty} O_{N,k}, \]

where

\[ O_{N,k} := \{ (f, g) \in H^s(\mathbb{T}^3) : \| \Phi(t)(\rho_{\epsilon_n} * (f, g)) \|_{L^\infty([0,1];H^s(\mathbb{T}^3))} > N \}. \]

- It is relatively straightforward to show that \( O_{N,k} \) open. Therefore \( O \) is a \( G_\delta \) set.
Open problems

- I would be interested if one can extend this kind of results to other equations.

- Our proof meets serious difficulties in the context of the nonlinear Schrödinger equation (not only because of the lack of finite propagation speed).
Related problems and results

- $2d$ low regularity weak solutions of the Euler equation as the ones obtained by De Lellis- Székelyhidi are *not limits* of the smooth solutions obtained by some regularisation of the data.
- $2d$ low regularity weak solutions of the $2d$ Euler equation with white noise vorticity obtained by Flandoli are limits of true smooth solutions of $2d$ Euler for some regularisation of the data.
- Low regularity solutions of KPZ obtained by Hairer are obtained as unique limits of smooth solutions of the equations with a suitably regularized noise. But in similar results for the heat equation this is not the case.
- Low regularity solutions of the Benjamin-Ono equation obtained by Ionescu-Kenig, Gérard-Kappeler-Topalov are obtained as *unique* limits of smooth solutions of the equation for every regularisation of the data.
Thank you for your attention!