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Concerning the pathological set in the context of probabilistic well-posedness

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(based joint work with Chenmin Sun)

The nonlinear wave equation

• In this talk, we consider the Cauchy problem for the nonlinear wave equation, posed on the $3d$ torus $\mathbb{T}^3$:

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \,. \qquad (1)$$

• The energy

$$\int_{\mathbb{T}^3} \Big((\partial_t u)^2 + |\nabla u|^2 \Big) + \frac{1}{2} \int_{\mathbb{T}^3} u^4$$

is formally conserved by (1). For $s \in \mathbb{R}$, we set

$$\mathcal{H}^s(\mathbb{T}^3) := H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$$

which is a natural phase space for (1).

Theorem 1 (classical)

• *For every $(u_0, u_1) \in \mathcal{H}^1(\mathbb{T}^3)$ there exists a unique global solution of (1) in the class $(u, \partial_t u) \in C(\mathbb{R}; \mathcal{H}^1(\mathbb{T}^3))$.*

• *If in addition $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$ for some $s \geq 1$ then*

$$(u, \partial_t u) \in C(\mathbb{R}; \mathcal{H}^s(\mathbb{T}^3)).$$

• *The dependence with respect to the initial data is continuous.*

Lower regularity using the dispersion

- Thanks to the Strichartz estimates the local in time part of Theorem 1 can be extended to the class $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$, $s \geq 1/2$.
- The global in time part can be extended to $s > 3/4$ using almost conservation law techniques :

Theorem 2 (Kenig-Ponce-Vega, Gallagher-Planchon, Roy)
Let $s > 3/4$ and fix $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$. Let $(u_{0,n}, u_{1,n})_{n\in\mathbb{N}}$ be **any sequence** *of smooth data approximating (u_0, u_1) in $\mathcal{H}^s(\mathbb{T}^3)$ and let $u_n(t,x)$ be the smooth solution of*

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u|_{t=0} = u_{0,n},, \quad \partial_t u|_{t=0} = u_{1,n}.$$

Then there exists a limit object $u(t)$ such that for any $T > 0$,

$$\lim_{n\to\infty} \Big\| (u_n(t), \partial_t u_n(t)) - (u(t), \partial_t u(t)) \Big\|_{L^\infty([-T,T];\mathcal{H}^s(\mathbb{T}^3))} = 0.$$

Moreover $u(t)$ solves the nonlinear wave equation in the sense of distributions.

- We conjecture that Theorem 2 remains true for $s \geq 1/2$ (proved recently by Dodson in the radial case of $\mathbb{R}^3$).

$\boxed{\text{Limit of the deterministic methods}}$

Theorem 3

Let $s \in (0, 1/2)$ et $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3)$. There exists a sequence

$$u_N(t,x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \qquad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta)u_N + u_N^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N(0) - u_0, \partial_t u_N(0) - u_1)\|_{\mathcal{H}^s(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \|u_N(t)\|_{L^\infty([-T,T];H^s(\mathbb{T}^3))} = +\infty.$$

• The proof is based on an idea introduced by **Gilles Lebeau** and further developed by Christ-Colliander-Tao, Burq-Gérard–Tz., Xia.

$\boxed{\text{Solving the equation by probabilistic methods}}$

• We can ask whether some form of well-posedness survives for initial data in $\mathcal{H}^s(\mathbb{T}^3), s < 1/2$?

• The answer of this question is positive if we endow the space $\mathcal{H}^s(\mathbb{T}^3), s < 1/2$ with a non degenerate probability measure such that we have the existence, the uniqueness, and a form of continuous dependence almost surely with respect to this measure.

Probabilistic global well-posedness

Theorem 4 (Burq-Tz. 2008)
Let $0 \leq s < \frac{1}{2}$. Then there is a dense set $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ satisfying $\Sigma \cap \mathcal{H}^{s'}(\mathbb{T}^3) = \emptyset$ for every $s' > s$ such that the following holds true. For every $(f, g) \in \Sigma$, denote by $(u_n(t))_{n\in\mathbb{N}}$ the smooth solutions of

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u(0, x) = \rho_n * f, , \quad \partial_t u(0, x) = \rho_n * g \,,$$

where $(\rho_n)_{n\in\mathbb{N}}$ is an approximate identity. Then there exists a limit object $u(t)$ such that for any $T > 0$,

$$\lim_{n\to\infty} \Big\|(u_n(t), \partial_t u_n(t)) - (u(t), \partial_t u(t))\Big\|_{L^\infty([-T,T];\mathcal{H}^s(\mathbb{T}^3))} = 0.$$

Moreover $u(t)$ solves the nonlinear wave equation in the distributional sense.

Comments

• The proof of the previous result is inspired by the seminal contribution of 1994 by **Bourgain**. There are however several new features :

• The first one is that more general randomisations are allowed. This led to similar results in the context of a non compact spatial domains.

• The argument allowing to pass from local to global solutions is based on a probabilistic energy estimates (Gronwall method) while the argument giving the globalisation of the local solutions in the Bourgain work is restricted to a very particular distribution of the initial data related to the Gibbs measure (measure preserving method).

• The result by Burq-Tz. deals with functions of positive Sobolev regularity which avoids a renormalization of the equation, making the results more natural from a purely PDE perspective. In particular, it is clear what ill-posedness means.

$\boxed{\text{Comments (sequel)}}$

• It would very interesting to prove such type of results for quasi-linear wave equations.

• A first result in this direction was obtained by B. Bringmann who proved probabilistic well-posedness for equations of type

$$(\partial_t^2 - \Delta)u = F(\nabla u),$$

and a quadratic F.

The pathological set

• The result by Burz-Tz. provides a nice dense set Σ of initial data such that for good approximations we get nice global solutions (but for bad approximations we get divergent sequences !).

• What about initial data outside Σ ?

Theorem 5 (Sun-Tz. 2020)

Let $0 < s < \frac{1}{2}$. Then there is an approximate identity $(\rho_n)_{n\in\mathbb{N}}$ and there is a dense set $S \subset \mathcal{H}^s(\mathbb{T}^3)$ such that for every $(f,g) \in S$, the family of the smooth solutions of

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u(0,x) = \rho_n * f, \quad \partial_t u(0,x) = \rho_n * g$$

do not converge. More precisely, for every $T > 0$,

$$\lim_{n\to\infty} \|u_n(t)\|_{L^\infty([0,T];H^s(\mathbb{T}^3))} = +\infty\,.$$

The pathological set (sequel)

• Consider again

$$(\partial_t^2 - \Delta)u_n + u_n^3 = 0, \quad u(0,x) = \rho_n * f, \quad \partial_t u(0,x) = \rho_n * g \qquad (2)$$

and let $\mathcal{P}$ be the set of $(f,g) \in \mathcal{H}^s(\mathbb{T}^3)$ such that the solution u_n of (2) satisfies the property

$$\limsup_{n\to\infty} \|u_n(t)\|_{L^\infty([0,1];H^s(\mathbb{T}^3))} = +\infty \, .$$

Corollary 6
The set $\mathcal{P}$ contains a dense G_δ subset of $\mathcal{H}^s(\mathbb{T}^3)$.

• Consequently, by the Baire category theorem, the good data set Σ in the Burq-Tz. theorem is not a G_δ subset of $\mathcal{H}^s(\mathbb{T}^3)$.
• On the other hand, the pathological sets are negligible with respect to the measures introduced by the natural gaussian fields used in the probabilistic well-posedness results.

Comments

• The previous discussion confirms that the topological and the measure theoretic notions of genericity are very different.

• For examples of G_δ dense sets giving solutions of Hamiltonian PDE's with growing Sobolev norms for large times, we refer to the works by Hani and Grellier-Gérard.

• In our result, the Sobolev norms are growing in very short times, depending on the frequency localization of the initial data.

• Naive question : Is such a phenomenon present in the context of the Lindblad ill-posedness results for quasi-linear wave equations ?

The ODE profile

• The basic idea is that since the regularity is supercritical the linear part of the equation is treated as a perturbation.
• Therefore, we consider the solution of the ODE

$$V'' + V^3 = 0, \quad V(0) = 1,\ V'(0) = 0$$

which is globally defined and periodic (oscillating between 0 and 1).
• Let us fix the the positive bump functions $\rho, \varphi \in C^\infty(\mathbb{R}^3)$, supported in $|x| \leq \frac{1}{100}$, seen as functions on $\mathbb{T}^3$ with $\int \rho = 1$. As usual, $\rho_\epsilon(x) := \epsilon^{-3}\rho(x/\epsilon)$. Let

$$v_n(0,x) := \kappa_n n^{\frac{3}{2}-s}\varphi(nx), \quad v_n^\epsilon(0,x) := \rho_\epsilon * v_n(0,x),\ \kappa_n = (\log n)^{-\delta_1},\ \delta_1 > 0\,.$$

• Define

$$v_n^\epsilon(t,x) = v_n^\epsilon(0,x)V(t\,v_n^\epsilon(0,x)).$$

Then one verifies that v_n^ϵ solves the dispersionless equation

$$\partial_t^2 v_n^\epsilon + (v_n^\epsilon)^3 = 0, \quad (v_n^\epsilon, \partial_t v_n^\epsilon)|_{t=0} = (v_n^\epsilon(0,x), 0).$$

Control on the profile

Proposition 7
Let $0 < s < \frac{1}{2}$. *Set*

$$\epsilon_n = \frac{1}{100n}, \quad t_n = (\log n)^{\delta_2} n^{-\left(\frac{3}{2}-s\right)}, \quad \delta_2 > 0.$$

Then we have the lower bound

$$\|v_n^{\epsilon_n}(t_n)\|_{H^s(\mathbb{T}^3)} \gtrsim \kappa_n (\log n)^{(\delta_2-\delta_1)s}$$

and the upper bounds

$$\|v_n^{\epsilon_n}(t)\|_{H^k(\mathbb{T}^3)} \lesssim \kappa_n \, (\log n)^{(\delta_2-\delta_1)k} \, n^{k-s}, \quad k = 0, 1, 2, 3, \cdots, \quad t \in [0, t_n],$$

$$\|\partial^\alpha v_n^{\epsilon_n}(t)\|_{L^\infty(\mathbb{T}^3)} \lesssim (\log n)^{\delta_2-\delta_1} \, n^{|\alpha|} \, \kappa_n \, n^{\frac{3}{2}-s}, \quad \alpha \in \mathbb{N}^3, |\alpha| = 0, 1, \quad t \in [0, t_n] \, .$$

The perturbative analysis

- For $(u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$, denote by $u_n^{\epsilon_n}$ the solution of

$$\partial_t^2 u_n^{\epsilon_n} - \Delta u_n^{\epsilon_n} + (u_n^{\epsilon_n})^3 = 0$$

with the initial data

$$(u_n^{\epsilon_n}(0), \partial_t u_n^{\epsilon_n}(0)) = \rho_{\epsilon_n} * \Big((u_0, u_1) + (v_n(0), 0)\Big).$$

Proposition 8

Assume that $0 < s < \frac{1}{2}$. *Then for any* $0 < \theta < \frac{1}{2}\Big(\frac{1}{2} - s\Big)$ *there exist* $C > 0$, $\delta_2 > 0$, *such that for any* $\delta_1 \in (0, \delta_2)$, *we have*

$$\sup_{t \in [0, t_n]} \|u_n^{\epsilon_n}(t) - \rho_{\epsilon_n} * S(t)(u_0, u_1) - v_n^{\epsilon_n}(t)\|_{H^\nu(\mathbb{T}^3)} \leq C n^{(\nu - s) - \theta}, \forall \nu = 0, 1, 2,$$

where $S(t)$ *is the free evolution and the constant* C *only depends on the smooth data* (u_0, u_1) *and* $\theta > 0$. *Consequently, we have*

$$\sup_{t \in [0, t_n]} \|u_n^{\epsilon_n}(t) - \rho_{\epsilon_n} * S(t)(u_0, u_1) - v_n^{\epsilon_n}(t)\|_{H^s(\mathbb{T}^3)} \leq C n^{-\theta}.$$

In particular, for δ_1 *sufficiently small,*

$$\|u_n^{\epsilon_n}(t_n)\|_{H^s(\mathbb{T}^3)} \gtrsim (\log n)^{s(\delta_2 - \delta_1) - \delta_1} \to \infty, \text{ as } n \to \infty.$$

$\boxed{\text{Finite propagation speed of the nonlinear wave equation}}$

Proposition 9

Let w_1, w_2 *be two* C^∞ *solutions of the nonlinear wave equation*

$$\partial_t^2 w - \Delta w + w^3 = 0.$$

If the initial data $(w_1(0), \partial_t w_1(0)), (w_2(0), \partial_t w_2(0))$ *coincide on the ball*

$$B(x_0, r_0) := \{x \in \mathbb{R}^3 : |x - x_0| \leq r_0\}$$

then for $0 \leq t < r_0$,

$$(w_1(t), \partial_t w_1(t)) = (w_2(t), \partial_t w_2(t))$$

on $B(x_0, r_0 - t)$.

Proof of the main result

• We identify $\mathbb{T}^3$ with $[-\pi,\pi]^3$ and we use the coordinate system $x=(x_1,x')$ near the origin. Let $z^k=(\frac{1}{k},0,0)$

• Let $n_k=\mathrm{e}^{\mathrm{e}^k}$, and define

$$v_{0,k}(x):=(\log n_k)^{-\delta_1}n_k^{\frac{3}{2}-s}\varphi(n_k(x_1-\frac{1}{k}),n_kx')=v_{n_k}(0,\cdot-z^k)\,.$$

There exists k_0, such that for all $k\geq k_0$, the supports of $v_{0,k}$ are pairwise disjoint and for $k_0\leq k_1<k_2$,

$$\mathrm{dist}\Big(\mathrm{supp}(v_{0,k_1}),\mathrm{supp}(v_{0,k_2})\Big)\sim\frac{1}{k_1}-\frac{1}{k_2}.$$

• Denote by $B_k=B(z^k,r_k)$, where $r_k=\frac{1}{k^3}$. For $k_0\gg 1$, the balls $B_k,k\geq k_0$ are mutually disjoint. Moreover,

$$\mathrm{supp}(\rho_{\epsilon_{n_k}}*v_{0,k})\subset\widetilde{B}_k,$$

where $\widetilde{B}_k=B(z^k,r_k/3)$ (recall that $\epsilon_{n_k}=\frac{1}{100n_k}$).

Proof of the main result (sequel)

- We have that

$$\mathrm{dist}\Big(\mathrm{supp}(\rho_{\epsilon_{n_k}} * (v_0 - v_{0,k})), B_k\Big) \gtrsim \frac{1}{k^2},$$

where

$$v_0 = \sum_{k \geq k_0} v_{0,k} \in H^s(\mathbb{T}^3).$$

- In particular, for any $(u_0, u_1) \in C^\infty \times C^\infty$,

$$\rho_{\epsilon_{n_k}} * ((u_0, u_1) + (v_0, 0))$$

coincides with

$$\rho_{\epsilon_{n_k}} * ((u_0, u_1) + (v_{0,k}, 0))$$

on the ball B_k.

Proof of the main result (definition of the pathological set)

- Let $k_0 \gg 1$, defined in the previous discussion. Set

$$S = C^{\infty}(\mathbb{T}^3) \times C^{\infty}(\mathbb{T}^3) + \Big\{ \Big(\sum_{k=k_1}^{\infty} v_{0,k}, 0 \Big) : k_1 \geq k_0 \Big\}.$$

- Since

$$\Big\| \sum_{k=k_1}^{\infty} v_{0,k} \Big\|_{H^s(\mathbb{T}^3)} \leq \sum_{k=k_1}^{\infty} \|v_{0,k}\|_{H^s(\mathbb{T}^3)} \leq \sum_{k=k_1}^{\infty} \mathrm{e}^{-k\delta_1} \to 0 \text{ as } k_1 \to \infty,$$

we conclude that S is dense in $\mathcal{H}^s(\mathbb{T}^3)$.

Proof of the main result (sequel)

• Fix $(f, g) \in S$. By definition, there exists $(u_0, u_1) \in C^\infty \times C^\infty$ and $k_1 \geq k_0$, such that

$$(f, g) = (u_0, u_1) + \Big(\sum_{k=k_1}^{\infty} v_{0,k}, 0 \Big).$$

• Our goal is to show that, for any $N > 0$ and any $\delta > 0$, there exist $\tau_N \in [0, T]$ and $0 < \epsilon < \delta$, such that the solution u^ϵ to our equation with initial data $\rho_\epsilon * (f, g)$ satisfies

$$\|u^\epsilon(\tau_N)\|_{H^s(\mathbb{T}^3)} > N. \tag{3}$$

• We will choose $k \geq k_1$, large enough, such that

$$\kappa_{n_k} (\log n_k)^{(\delta_2 - \delta_1)s} \gtrsim N, \quad \epsilon_k = \frac{1}{100 n_k} < \delta.$$

This can be achieved by choosing $\delta_1 < \delta_2$ such that $s(\delta_2 - \delta_1) > \delta_1$.

Proof of the main result (sequel)

• Let $\widetilde{u}_k$ be the solution of our equation with the initial data

$$\rho_{\epsilon_{n_k}} * (u_0, u_1) + \rho_{\epsilon_{n_k}} * (v_{0,k}, 0)\,.$$

• Let $\widetilde{v}_k$ be the solution of

$$\partial_t^2 \widetilde{v_k} + (\widetilde{v_k})^3 = 0$$

with the initial data $\rho_{\epsilon_{n_k}} * (v_{0,k}, 0)$.

• We remark that $\widetilde{v}_k, \widetilde{u}_k$ are just $v_{n_k}^{\epsilon_{n_k}}, u_{n_k}^{\epsilon_{n_k}}$ the proposition of the perturbative analysis, up to translation.

• In particular,

$$\|\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} \gtrsim (\log n_k)^{s(\delta_2-\delta_1)-\delta_1}, \tag{4}$$

and

$$\|\widetilde{u}_k(t_{n_k}) - \rho_{\epsilon_{n_k}} * S(t_{n_k})(u_0, u_1) - \widetilde{v}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} \lesssim n_k^{-\theta}. \tag{5}$$

Proof of the main result (sequel)

• We now apply the finite propagation speed property to $\widetilde{u}_k$ and $u^{\epsilon_{n_k}}$. At $t=0$,

$$(u^{\epsilon_{n_k}}(0), \partial_t u^{\epsilon_{n_k}}(0))|_{B_k} = (\widetilde{u}_k(0), \partial_t \widetilde{u}_k(0))|_{B_k},$$

and therefore

$$(u^{\epsilon_{n_k}}(t), \partial_t u^{\epsilon_{n_k}}(t))|_{B(z^k, r_k - t)} = (\widetilde{u}_k(t), \partial_t \widetilde{u}_k(t))|_{B(z^k, r_k - t)}, \quad \forall 0 \leq t < r_k.$$

• In particular, for large k,

$$(u^{\epsilon_{n_k}}(t), \partial_t u^{\epsilon_{n_k}}(t))|_{B(z^k, r_k/2)} = (\widetilde{u}_k(t), \partial_t \widetilde{u}_k(t))|_{B(z^k, r_k/2)}, \quad \forall t \in [0, t_{n_k}].$$

• Take $\chi \in C_c^\infty(\mathbb{R}^3)$, such that $\chi(x) \equiv 1$ if $|x| < \frac{1}{3}$ and $\chi \equiv 0$ if $|x| \geq \frac{1}{2}$. Define $\chi_k(x) := \chi((x - z^k)/r_k)$, hence $\chi_k|_{\widetilde{B}_k} \equiv 1$ and $\chi_k|_{(B(z^k, r_k/2))^c} \equiv 0$. Therefore, we have

$$\chi_k(x)(u^{\epsilon_{n_k}}(t), \partial_t u^{\epsilon_{n_k}}(t)) = \chi_k(x)(\widetilde{u}_k(t), \partial_t \widetilde{u}_k(t)), \quad \forall t \in [0, t_{n_k}].$$

Proof of the main result (sequel)

- Since $s < 1/2$, we can localise

$$\|u^{\epsilon_{n_k}}(t_{n_k})\|_{H^s(\mathbb{T}^3)} \gtrsim \|\chi_k u^{\epsilon_{n_k}}(t_{n_k})\|_{H^s(\mathbb{T}^3)} \sim \|\chi_k(x)\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)}.$$

- Next, we can write

$$\begin{aligned}\|\chi_k(x)\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} \geq & \|\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} - \|(1-\chi_k)\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} \\ = & \|\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} - \|(1-\chi_k)(\widetilde{u}_k(t_{n_k}) - \widetilde{v}_k(t_{n_k}))\|_{H^s(\mathbb{T}^3)},\end{aligned}$$

where in the last equality, we crucially use the fact that

$$(1-\chi_k)\widetilde{v}_k(t_{n_k}) = 0,$$

thanks to the support property of $\widetilde{v}_k$.

Proof of the main result (sequel)

- Therefore, we have

$$
\begin{aligned}
\|u^{\epsilon_{n_k}}(t_{n_k})\|_{H^s(\mathbb{T}^3)} \gtrsim & \|\widetilde{u}_k(t_{n_k})\|_{H^s(\mathbb{T}^3)} - \|(1-\chi_k)(\rho_{\epsilon_{n_k}} * S(t_{n_k})(u_0,u_1))\|_{H^s(\mathbb{T}^3)} \\
& - \|(1-\chi_k)\Big(\widetilde{u}_k(t_{n_k}) - \rho_{\epsilon_{n_k}} * S(t_{n_k})(u_0,u_1) - \widetilde{v}_k(t_{n_k})\Big)\|_{H^s(\mathbb{T}^3)}.
\end{aligned}
$$

- Consequently

$$
\|u^{\epsilon_{n_k}}(t_{n_k})\|_{H^s(\mathbb{T}^3)} \gtrsim (\log n_k)^{s(\delta_2-\delta_1)-\delta_1} - C - n_k^{-\theta}.
$$

- It remains to choose $\delta_1 > 0$ small such that $s(\delta_2 - \delta_1) - \delta_1 > 0$ and $k \gg 1$.

- This completes the proof.

On the proof of the corollary

• Denote by $u^\epsilon(t) = \Phi(t)(\rho_\epsilon * (f,g))$ the solution of the cubic wave equation with initial data $\rho_\epsilon * (f,g)$.

• The set

$$\mathcal{O} := \{(f,g) \in \mathcal{H}^s(\mathbb{T}^3) : \limsup_{k\to\infty} \|\Phi(t)(\rho_{\epsilon_{n_k}} * (f,g))\|_{L^\infty([0,1];H^s(\mathbb{T}^3))} = \infty\}$$

is contained in the pathological set $\mathcal{P}$.

• As a byproduct of the previous analysis, $S \subset \mathcal{O}$, hence $\mathcal{O}$ is dense.

• In addition

$$\mathcal{O} = \bigcap_{N=1}^{\infty} \mathcal{O}_N,$$

where

$$\mathcal{O}_N := \{(f,g) \in \mathcal{H}^s(\mathbb{T}^3) : \limsup_{k\to\infty} \|\Phi(t)(\rho_{\epsilon_{n_k}} * (f,g))\|_{L^\infty([0,1];H^s(\mathbb{T}^3))} > N\}.$$

On the proof of the corollary (sequel)

- By definition,

$$\mathcal{O}_N = \bigcap_{k_0=1}^{\infty} \bigcup_{k=k_0} \mathcal{O}_{N,k},$$

where

$$\mathcal{O}_{N,k} := \{(f,g) \in \mathcal{H}^s(\mathbb{T}^3) : \|\Phi(t)(\rho_{\epsilon_{n_k}} * (f,g))\|_{L^\infty([0,1];H^s(\mathbb{T}^3))} > N\}.$$

- It is relatively straightforward to show that $\mathcal{O}_{N,k}$ open. Therefore $\mathcal{O}$ is a G_δ set.

$\boxed{\text{Open problems}}$

• I would be interested if one can extend this kind of results to other equations.

• Our proof meets serious difficulties in the context of the nonlinear Schrödinger equation (not only because of the lack of finite propagation speed).

Related problems and results

• $2d$ low regularity weak solutions of the Euler equation as the ones obtained by De Lellis- Székelyhidi are *not limits* of the smooth solutions obtained by some regularisation of the data.

• $2d$ low regularity weak solutions of the 2d Euler equation with white noise vorticity obtained by Flandoli are limits of true smooth solutions of 2d Euler for some regularisation of the data.

• Low regularity solutions of KPZ obtained by Hairer are obtained as unique limits of smooth solutions of the equations with a suitably regularized noise. But in similar results for the heat equation this is not the case.

• Low regularity solutions of the Benjamin-Ono equation obtained by Ionescu-Kenig, Gérard-Kappeler-Topalov are obtained as *unique* limits of smooth solutions of the equation for *every* regularisation of the data.

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Thank you for your attention !