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On the propagation of gaussian measures under the flow of Hamiltonian PDE's

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$\boxed{\text{Multiple Fourier series}}$

- Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be a torus of dimension d.
- If $f : \mathbb{T}^d \to \mathbb{C}$ is a C^∞ function then for every $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

where $\widehat{f}(n)$ are the Fourier coefficients of f, defined by

$$\widehat{f}(n) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx\,.$$

- In particular

$$\widehat{f}(0) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)\, dx\,.$$

Sobolev spaces on the torus

• For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we set

$$\langle x \rangle := (1 + x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}.$$

• For $s \in \mathbb{R}$, we define the Sobolev norm of f by

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\widehat{f}(n)|^2. \tag{1}$$

• For $s \geq 0$ an integer, we have the norm equivalence

$$\|f\|^2_{H^s(\mathbb{T}^d)} \approx \sum_{|\alpha| \leq s} \|\partial^\alpha f\|^2_{L^2(\mathbb{T}^d)}. \tag{2}$$

In (2), ∂^α denotes a partial derivative of order at most s.

• For $s = 0$, we recover the Lebesgue space $L^2(\mathbb{T}^d)$.

• The Sobolev space $H^s(\mathbb{T}^d)$ is defined as the closure of $C^\infty(\mathbb{T}^d)$ with respect to the norm (1).

Sobolev spaces on a manifold

• Let (M,g) be a compact riemannian boundaryless manifold and $(\varphi_n)_{n\geq 0}$ be an orthonormal basis of $L^2(M)$ diagonalizing the Laplace-Beltrami operator Δ_g such that

$$-\Delta_g\varphi_n = \lambda_n^2\varphi_n,$$

with $0=\lambda_0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots$.

• Then the Sobolev norm of a smooth function f on M is defined by

$$\|f\|^2_{H^s(M)} = \sum_{n=0}^{\infty}\langle\lambda_n\rangle^{2s}|\widehat{f}(n)|^2\,, \tag{3}$$

where

$$\widehat{f}(n)=\int_M f(x)\overline{\varphi_n(x)}dx$$

is the "size" of the projection of f on the line spanned by φ_n.

• The Sobolev space $H^s(M)$ is again defined as the closure of $C^\infty(M)$ with respect to the norm (3).

Almost sure improvements of the Sobolev embedding

• We now discuss *an almost sure improvement* of the Sobolev embedding.

• We say that a random variable g belongs to $\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$, if $g=h+il$, where $h\in\mathcal{N}(0,\sigma^2)$ and $l\in\mathcal{N}(0,\sigma^2)$ are independent.

• Let $u\in L^2(\mathbb{T})$ be a deterministic function. There is a sequence $(c_n)_{n\in\mathbb{Z}}\in l^2(\mathbb{Z})$ (the Fourier coefficients of u) such that

$$u(x)=\sum_{n\in\mathbb{Z}} c_n e^{inx}\,.$$

• Consider now a randomised version of u given by the expression

$$u_\omega(x)=\sum_{n\in\mathbb{Z}} c_n\, g_n(\omega)\, e^{inx}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ are independent belonging to $\mathcal{N}_{\mathbb{C}}(0,1)$.

Almost sure improvements of the Sobolev embedding (sequel)

• We have $g_n(\omega)\, e^{inx} \in \mathcal{N}_{\mathbb{C}}(0,1)$ (invariance under rotations of $\mathcal{N}_{\mathbb{C}}(0,1)$).

• Next, using the independence of g_n we get that for a fixed $x \in \mathbb{T}$,

$$u_\omega(x) \in \mathcal{N}_{\mathbb{C}}\Big(0, \sum_{n\in\mathbb{Z}} |c_n|^2\Big).$$

• Thus $\forall\, p<\infty$, $\|u_\omega(x)\|_{L^p(\Omega)}$ is finite and independent of x. Consequently $u_\omega(x) \in L^p(\Omega\times\mathbb{T})$ which thanks fo Fubini gives :

Proposition 1

For every $p<\infty$, $u_\omega(x)\in L^p(\mathbb{T})$, almost surely.

• The last statement is to compare with the Sobolev embedding : $H^{\frac{1}{2}}(\mathbb{T})$ *is continuously embedded in* $L^p(\mathbb{T})$ *for every* $p<\infty$. *The statement is false, if we replace* $H^{\frac{1}{2}}(\mathbb{T})$ *with* $H^s(\mathbb{T})$ *for some* $s<1/2$.

Almost sure improvements of the Sobolev embedding (sequel)

• Very informally : **the randomisation gains a $1/2$ derivative.**

• We can replace the gaussians with much more general random variables. For instance, let

$$v_\omega(x) = \sum_{n\in\mathbb{Z}} c_n\, h_n(\omega)\, e^{inx}\,,$$

where $(h_n(\omega))_{n\in\mathbb{Z}}$ are independent standard Bernoulli random variables (random signs). Then we have :

Proposition 2

For every $p<\infty$, $v_\omega(x)\in L^p(\mathbb{T})$, almost surely.

Remark. The random function $v_\omega(x)$ and the deterministic function

$$v(x) = \sum_{n\in\mathbb{Z}} c_n\, e^{inx}$$

have Fourier coefficients with the same absolute value.

Almost sure improvements of the Sobolev embedding (sequel)

• The key point in the case of Bernoulli random variables is the large deviation bound (based on the exponential method)

$$p\Big(\omega \, : \, \Big| \sum_{n\in\mathbb{Z}} c_n \, h_n(\omega) \, e^{inx} \Big| > \lambda\Big) \leq C \exp^{-\frac{c\lambda^2}{\sum_n |c_n|^2}} ,$$

where C and c are positive constants independent of $x \in \mathbb{T}$.

• The large deviation bound in turn implies

$$\Big\| \sum_{n\in\mathbb{Z}} c_n \, h_n(\omega) \, e^{inx} \Big\|_{L^p(\Omega)} \leq C\sqrt{p}\Big(\sum_{n\in\mathbb{Z}} |c_n|^2 \Big)^{\frac{1}{2}},$$

with a constant C independent of x.

• With the last bound in hand, we can proceed as in the gaussian case.

• • Let again $u \in L^2(\mathbb{T})$ be a deterministic function. Suppose that $u \notin H^s(\mathbb{T})$ for some $s > 0$.

• Consider again the randomised version of u given by

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \widehat{u}(n)\, g_n(\omega)\, e^{inx}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ are independent from $\mathcal{N}_{\mathbb{C}}(0,1)$.

Proposition 3

$u_\omega(x) \notin H^s(\mathbb{T})$, *almost surely.*

Proof. We have that the event

$$\{\omega\,:\, \|u_\omega\|_{H^s} < \infty\}$$

belongs to the asymptotic σ-algebra obtained from the independent σ-algebras generated from g_n because the property $\|u\|_{H^s} < \infty$ depends only on $(1 - \Pi_N)u$ for every $N \in \mathbb{N}$, where Π_N is the Dirichlet projector.

No improvement of the Sobolev regularity (sequel of the proof)

• Therefore by the Kolmogorov zero-one law, we have that

$$p(\{\omega : \|u_\omega\|_{H^s} < \infty\}) \in \{0,1\}.$$

• We suppose that the last probability is 1 and we look for a contradiction. If the probability is one then by the dominated convergence almost surely

$$\lim_{N\to\infty} \int_\Omega e^{-\|\pi_N u_\omega\|^2_{H^s}}\, dp(\omega) = \int_\Omega e^{-\|u_\omega\|^2_{H^s}}\, dp(\omega) > 0\,. \tag{4}$$

• We will show that

$$\lim_{N\to\infty} \int_\Omega e^{-\|\pi_N u_\omega\|^2_{H^s}}\, dp(\omega) = 0$$

which will be in a contradiction with (4).

• Using the independence, we can write

$$\int_\Omega e^{-\|\pi_N u_\omega\|^2_{H^s}}\, dp(\omega) = \prod_{|n|\leq N} \int_{\mathbb{R}^2} e^{-|\widehat{u}(n)|^2(x^2+y^2)\langle n\rangle^{2s}} e^{-(x^2+y^2)} \frac{dxdy}{\pi}\,.$$

No improvement of the Sobolev regularity (sequel of the proof)

- Now, if we set

$$\theta := \int_{x^2+y^2\leq 1} e^{-(x^2+y^2)}\frac{dxdy}{\pi} \in (0,1)$$

and we have

$$\int_{\mathbb{R}^2} e^{-|\widehat{u}(n)|^2(x^2+y^2)\langle n\rangle^{2s}} e^{-(x^2+y^2)}\frac{dxdy}{\pi}$$
$$\leq \theta + \int_{x^2+y^2>1} e^{-|\widehat{u}(n)|^2\langle n\rangle^{2s}} e^{-(x^2+y^2)}\frac{dxdy}{\pi}$$
$$\leq \theta + e^{-|\widehat{u}(n)|^2\langle n\rangle^{2s}}(1-\theta) = 1-(1-\theta)(1-e^{-|\widehat{u}(n)|^2\langle n\rangle^{2s}})\,.$$

- Now, we observe that

$$\lim_{N\to\infty}\sum_{|n|\leq N}(1-e^{-|\widehat{u}(n)|^2\langle n\rangle^{2s}}) = \infty$$

because

$$\lim_{N\to\infty}\sum_{|n|\leq N}\langle n\rangle^{2s}|\widehat{u}(n)|^2 = \infty$$

by assumption. This completes the proof.

- Consider the random series :

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha}\, e^{inx}, \qquad \frac{1}{4} < \alpha < \frac{1}{2},$$

with g_n as in the previous discussion.

Proposition 4

We have that a.s. $u_\omega \in H^\sigma(\mathbb{T})$, $\sigma < \alpha - \frac{1}{2}$ *but a.s.* $u_\omega \notin H^{\alpha-\frac{1}{2}}(\mathbb{T})$.

Proof. We can write for $N < M$

$$\Big\| \sum_{N\leq|n|\leq M} e^{inx}\, \frac{g_n(\omega)}{\langle n\rangle^\alpha} \Big\|^2_{L^2(\Omega; H^\sigma(\mathbb{T}))} \simeq \sum_{N\leq|n|\leq M} \frac{\langle n\rangle^{2\sigma}}{\langle n\rangle^{2\alpha}}$$

which tends to zero as $N \to \infty$, provided

$$\sigma < \alpha - \frac{1}{2}\,.$$

This completes the proof.

Products in Sobolev spaces of negative indexes (sequel)

The random distribution

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha}\, e^{inx}, \qquad \frac{1}{4} < \alpha < \frac{1}{2}$$

belongs to a Sobolev space of negative regularity and therefore it is hard to define an object like $|u_\omega(x)|^2$. For example, thanks to Parseval, the zero Fourier coefficient of $|u_\omega(x)|^2$ should be

$$\sum_{n\in\mathbb{Z}} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}}$$

which is a.s. divergent. However, it turns out that the zero Fourier coefficient is the only obstruction and it is possible, *after a renormalisation*, to define $|u_\omega|^2$ and even to compute its Sobolev regularity.

Products in Sobolev spaces of negative indexes

• Fix $\sigma < \alpha - \frac{1}{2}$ (close to $\alpha - \frac{1}{2}$). Consider the partial sums

$$u_{\omega,N}(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^{\alpha}}\, e^{inx} \in C^{\infty}(\mathbb{T})$$

and write

$$|u_{\omega,N}(x)|^2 = \sum_{|n|\leq N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2 \\ |n_1|,|n_2|\leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}}\, e^{i(n_1-n_2)x}.$$

• The first term (the zero Fourier coefficient) contains all the singularity while the second has an a.s. limit in $H^{2\sigma}(\mathbb{T})$.

Products in Sobolev spaces of negative indexes

- Consequently, we set

$$c_N := \mathbb{E}\Big(\sum_{|n|\leq N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}}\Big) = \mathbb{E}(|u_{\omega,N}(x)|^2) = \sum_{|n|\leq N} \frac{2}{\langle n\rangle^{2\alpha}} \sim N^{1-2\alpha},$$

and we define the renormalised partial sums

$$|u_{\omega,N}(x)|^2 - c_N = \sum_{|n|\leq N} \frac{|g_n(\omega)|^2-2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}}\, e^{i(n_1-n_2)x}.$$

- Thanks to the independence of g_n we have

$$\mathbb{E}\Big(\Big| \sum_{|n|\leq N} \frac{|g_n(\omega)|^2-2}{\langle n\rangle^{2\alpha}}\Big|^2\Big) = \sum_{|n|\leq N} \frac{4}{\langle n\rangle^{4\alpha}},$$

which has a limit as $N\to\infty$ when $\alpha > 1/4$.

• Another use of the independence yields that

$$\mathbb{E}\Big(\Big\| \sum_{\substack{n_1 \neq n_2 \\ |n_1|,|n_2| \leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}}\, e^{i(n_1-n_2)x}\Big\|^2_{H^{2\sigma}}\Big)$$

is bounded by

$$C\sum_{n_1,n_2} \frac{\langle n_1-n_2\rangle^{4\sigma}}{\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}}\,.$$

The last sum is convergente as far as $-4\sigma+4\alpha > 2$, which is equivalent to our assumption $\sigma < \alpha - \frac{1}{2}$. Hence we proved that

Proposition 5

The sequence

$$\Big(|u_{\omega,N}(x)|^2 - c_N\Big)_{N\geq 1}$$

has a limit in $L^2(\Omega; H^{2\sigma}(\mathbb{T}))$. This limit is by definition the renormalisation of $|u_\omega|^2$.

Remarks

• Using more involved arguments, we can also show the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T})$ of the sequence

$$\Big(|u_{\omega,N}(x)|^2 - c_N\Big)_{N\geq 1}\,.$$

• Since $\sigma < 0$ the norm in $H^{2\sigma}(\mathbb{T})$ is weaker than in $H^{\sigma}(\mathbb{T})$ (where $u_\omega(x)$ is defined).

• Informally : the square of the modulus of an element of H^σ is in $H^{2\sigma}$, after a renormalisation.

• This is a remarkable probabilistic phenomenon, in the heart of the study of evolution partial differential equations in the presence of randomness in Sobolev spaces of negative indexes.

Remarks (sequel)

• Again, we can replace the gaussians with much more general random variables.

• We can also replace the sequence

$$\frac{1}{\langle n\rangle^{\alpha}}$$

with a more general sequence (c_n), i.e. we can consider

$$\sum_{n\in\mathbb{Z}} c_n\, g_n(\omega)\, e^{inx}$$

instead of

$$\sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^{\alpha}}\, e^{inx}$$

but I am not aware of the optimal regularity of the renormalised square in function of the sequence (c_n).

Products in Sobolev spaces of negative indexes on manifolds

• Let (M, g) be a compact riemannian boundaryless manifold of dimension 2 and $(\varphi_n)_{n\geq 0}$ be an orthonormal basis of the Laplace-Beltrami operator $-\Delta_g$ with corresponding increasing eigenvalues $(\lambda_n^2)_{n\geq 0}$.
• We suppose that φ_n are real valued.
• Consider the random series :

$$u_\omega(x) = \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\langle \lambda_n \rangle} \varphi_n(x)$$

with g_n as in the previous discussion but this time real valued.
• The process $u_\omega(x)$ is essentially the Gaussian Free Field (GFF) on the manifold M.
• Again, we ask the question : Can we define $|u_\omega(x)|^2$?

Products in Sobolev spaces of negative indexes on manifolds

• Let us first compute the Sobolev regularity of the GFF. Thanks to the Weyl law, we know that $\lambda_n \sim n^{1/2}$ and therefore

$$\|u_\omega(x)\|^2_{L^2(\Omega;H^s(M))} = \sum_{n=0}^{\infty} \frac{\langle\lambda_n\rangle^{2s}}{\langle\lambda_n\rangle^2} \sim \sum_{n=1}^{\infty} n^{s-1} \,.$$

Therefore the GFF is missing L^2 and one may show that it misses a.s. L^1 too. It is however in the negative Sobolev spaces $H^s(M)$, $s < 0$.

• Consider again the truncated series

$$u_{\omega,N}(x) = \sum_{\lambda_n \leq N} \frac{g_n(\omega)}{\langle\lambda_n\rangle} \varphi_n(x)$$

which is almost surely a $C^\infty(M)$ function.

Products in Sobolev spaces of negative indexes on manifolds

• Write

$$|u_{\omega,N}(x)|^2 = \sum_{\lambda_n \le N} \frac{|g_n(\omega)|^2 |\varphi_n(x)|^2}{\langle \lambda_n \rangle^2} +$$

$$\sum_{\substack{n_1 \neq n_2 \\ \lambda_{n_1}, \lambda_{n_2} \le N}} \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{\langle \lambda_{n_1} \rangle \langle \lambda_{n_2} \rangle} \varphi_{n_1}(x) \varphi_{n_2}(x) := I_N + II_N .$$

• We have that II_N converges in $L^2(\Omega; H^s(M))$ for every $s < 0$ and again the diagonal term I_N contains the singular contribution.

Products in Sobolev spaces of negative indexes on manifolds

- We have that

$$\mathbb{E}(I_N) = \sum_{\lambda_n \le N} \frac{|\varphi_n(x)|^2}{\langle \lambda_n \rangle^2}$$

- Thanks to Hörmander (1968)

$$\sum_{\lambda_n \le \lambda} |\varphi_n(x)|^2 = c\lambda^2 + r(\lambda, x), \quad \sup_{\lambda \ge 1} \sup_{x \in M} \lambda^{-1} |r(\lambda, x)| < \infty.$$

- Therefore, if we set

$$C_N(x) = \sum_{\lambda_n \le N} \frac{|\varphi_n(x)|^2}{\langle \lambda_n \rangle^2}$$

then we have that there are two positive independent of x constants c_1 and c_2 such that

$$c_1 \log(N) \le C_N(x) \le c_2 \log(N).$$

Proposition 6
The sequence $(|u_{\omega,N}(x)|^2 - C_N(x))_{N\geq 1}$ *converges in* $L^2(\Omega; H^s(M))$, $s < 0$.

Proof. We first study the convergence of

$$II_N = \sum_{\substack{n_1 \neq n_2 \\ \lambda_{n_1}, \lambda_{n_2} \leq N}} \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{\langle \lambda_{n_1} \rangle \langle \lambda_{n_2} \rangle} \varphi_{n_1}(x) \varphi_{n_2}(x).$$

• We need to evaluate

$$\|(1-\Delta_g)^{s/2} II_N\|^2_{L^2(\Omega\times M)} = \int_M \sum_{\substack{n_1 \neq n_2 \\ \lambda_{n_1}, \lambda_{n_2} \leq N}} \frac{|(1-\Delta_g)^{s/2}(\varphi_{n_1}(x)\varphi_{n_2}(x))|^2}{\langle \lambda_{n_1} \rangle^2 \langle \lambda_{n_2} \rangle^2}$$

- Next, we can write

$$(1-\Delta_g)^{s/2}(\varphi_{n_1}(x)\varphi_{n_2}(x)) = \sum_{n=0}^{\infty} \langle\lambda_n\rangle^{2s}\varphi_n(x)\gamma(n,n_1,n_2),$$

where

$$\gamma(n,n_1,n_2) = \int_M \varphi_{n_1}(x)\varphi_{n_2}(x)\varphi_n(x).$$

- Therefore

$$\|(1-\Delta_g)^{s/2} II_N\|^2_{L^2(\Omega\times M)} = \sum_{n=0}^{\infty} \sum_{\substack{n_1\neq n_2 \\ \lambda_{n_1},\lambda_{n_2}\leq N}} \frac{|\gamma(n,n_1,n_2)|^2}{\langle\lambda_{n_1}\rangle^2\langle\lambda_{n_2}\rangle^2\langle\lambda_n\rangle^{\varepsilon}}$$

where $\varepsilon = -2s > 0$.

• After performing dyadic decompositions, we need to evaluate the quantity

$$Q := \sum_{\lambda_{n_1}\sim L_1} \sum_{\lambda_{n_2}\sim L_2} \sum_{\lambda_n \sim L} \frac{|\gamma(n,n_1,n_2)|^2}{\langle\lambda_{n_1}\rangle^2\langle\lambda_{n_2}\rangle^2\langle\lambda_n\rangle^\varepsilon}$$

where L_1, L_2 and L are dyadic integers.
• By a symmetry between L_1 and L_2, we can suppose that $L_1 \geq L_2$.
• If $L \leq L_1$, we write

$$L_1^2 L_2^2 L^\varepsilon \, Q \lesssim \sum_{\lambda_n\sim L} \sum_{\lambda_{n_2}\sim L_2} \int_M (\varphi_n(x)\varphi_{n_2}(x))^2 \, dx$$

and by the Hörmander theorem

$$L_1^2 L_2^2 L^\varepsilon \, Q \lesssim L^2 L_2^2$$

we can conclude because

$$\frac{L^2 L_2^2}{L_1^2 L_2^2 L^\varepsilon} \leq \frac{1}{L_1^\varepsilon}$$

is summable over the dyadic integers L, L_1, L_2 such that $L, L_2 \leq L_1$.

Products in Sobolev spaces of negative indexes on manifolds

- If $L \geq L_1$, we write

$$L_1^2 L_2^2 L^{\varepsilon}\, Q \lesssim \sum_{\lambda_{n_1} \sim L_1} \sum_{\lambda_{n_2} \sim L_2} \int_M (\varphi_{n_1}(x)\varphi_{n_2}(x))^2\, dx$$

and by the Hörmander theorem

$$L_1^2 L_2^2 L^{\varepsilon}\, Q \lesssim L_1^2 L_2^2$$

we can conclude because

$$\frac{L_1^2 L_2^2}{L_1^2 L_2^2 L^{\varepsilon}} \leq \frac{1}{L^{\varepsilon}}$$

is summable over the dyadic integers L, L_1, L_2 such that $L_1, L_2 \leq L$.

- This ends the analysis of II_N.

• In the study of the convergence of $I_N - C_N(x)$, we are reduced to the convergence of the quantity

$$\tilde{I}_N := \sum_{\lambda_n \le N} \frac{(|g_n(\omega)|^2 - 1)|\varphi_n(x)|^2}{\langle \lambda_n \rangle^2} .$$

We can write

$$\|\tilde{I}_N\|^2_{L^2(\Omega\times M)} = C \int_M \sum_{\lambda_n \le N} \frac{|\varphi_n(x)|^4}{\langle \lambda_n \rangle^4} .$$

• By the Hörmander theorem and the bound

$$\sup_{x\in M} |\varphi_n(x)| \le C\lambda_n^{\frac{1}{2}}$$

we get

$$\sum_{\lambda_n \sim L} \frac{|\varphi_n(x)|^4}{\langle \lambda_n \rangle^4} \lesssim L^{-4} L \sum_{\lambda_n \sim L} |\varphi_n(x)|^2 \lesssim L^{-1}$$

which concludes the evaluation of I_N. This completes the proof.

Theorem 7 (classical)

• *For every* $(u_0, u_1) \in H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ *there exists a unique global solution of*

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x)$$

in the class $(u, \partial_t u) \in C(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$.

• *If in addition* $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ *for some* $s \geq 1$ *then*

$$(u, \partial_t u) \in C(\mathbb{R}; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)).$$

The dependence with respect to the initial data is continuous.

• The local in time part of Theorem 7 can be extended to the case $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$, $s \geq 1/2$, and the global in time part to $s > 13/18$ (Kenig-Ponce-Vega, Gallagher-Planchon, Bahouri-Chemin, Roy).

• We conjecture that Theorem 7 remains true for $s \geq 1/2$ (proved recently by Dodson in the radial case of $\mathbb{R}^3$).

Limit of the deterministic methods

Theorem 8 (ill-posedness)
Let $s \in (0,1/2)$ *et* $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. *There exists a sequence*

$$u_N(t,x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \quad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta)u_N + u_N^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N(0) - u_0, \partial_t u_N(0) - u_1)\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for every $T > 0$*,*

$$\lim_{N \to +\infty} \sup_{0 \le t \le T} \|u_N(t)\|_{H^s(\mathbb{T}^3)} = +\infty.$$

• The proof is based on an idea introduced by Gilles Lebeau and further developed by Christ-Colliander-Tao, Burq-Tz., Xia.

$\boxed{\text{Solving the equation by probabilistic methods}}$

• We can ask whether some form of well-posedness survives for initial data in

$$H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3), \quad s < 1/2. \tag{5}$$

• The answer of this question is positive if we endow the space (5) with a non degenerate probability measure such that we have the existence, the uniqueness, and a form of continuous dependence almost surely with respect to this measure.

Choice of the measure

• We will choose the initial data among the realisations of the following random series

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\,, \qquad u_1^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,. \qquad (6)$$

Here $\{g_n\}_{n\in\mathbb{Z}^3}$ et $\{h_n\}_{n\in\mathbb{Z}^3}$ are two families of independent random variables conditioned by $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$, so that u_0^ω and u_1^ω are real valued.

• In addition, we suppose that for $n \neq 0$, g_n and h_n are complex gaussians from $\mathcal{N}_{\mathbb{C}}(0,1)$, and that g_0 and h_0 are standard real gaussians from $\mathcal{N}(0,1)$.

• We can define the measure μ_α as the image measure by the map

$$\omega \longmapsto (u_0^\omega, u_1^\omega)$$

• **Question :** μ_α is a measure on which space ?

On the gaussian measure μ_α

- We can write for $N < M$

$$\Big\| \sum_{N\leq|n|\leq M} e^{in\cdot x} \, \frac{g_n(\omega)}{\langle n\rangle^\alpha} \Big\|^2_{L^2(\Omega;H^s(\mathbb{T}^3))} \simeq \sum_{N\leq|n|\leq M} \frac{\langle n\rangle^{2s}}{\langle n\rangle^{2\alpha}}$$

which tends to zero as $N \to \infty$, provided

$$s < \alpha - \frac{3}{2}\,.$$

- Therefore

$$\sum_{n\in\mathbb{Z}^3} e^{inx} \, \frac{g_n(\omega)}{\langle n\rangle^\alpha} \in L^2(\Omega; H^s(\mathbb{T}^3))\,.$$

- A similar analysis applies to

$$\sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,.$$

$\boxed{\text{On the gaussian measure } \mu_\alpha \text{ (sequel)}}$

- We conclude that the map

$$\omega \longmapsto \Big(\sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x} \, , \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x} \Big)$$

defines a probability measure on $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$.

- Moreover

$$\mu_\alpha(H^{\alpha-\frac{3}{2}}(\mathbb{T}^3) \times H^{\alpha-\frac{5}{2}}(\mathbb{T}^3)) = 0 \, .$$

$\boxed{\text{Reformulation of the ill-posedness result}}$

Theorem 9

Let $\alpha \in (3/2, 2)$ and $0 < s < \alpha - 3/2$. For almost every ω, there exists a sequence

$$u_N^\omega(t,x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \qquad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta)u_N^\omega + (u_N^\omega)^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N^\omega(0) - u_0^\omega, \partial_t u_N^\omega(0) - u_1^\omega)\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \sup_{0 \le t \le T} \|u_N^\omega(t)\|_{H^s(\mathbb{T}^3)} = +\infty.$$

We can however prove the following result:

Theorem 10 (Burq-Tz. (2010))
Let $\alpha \in (3/2, 2)$ and $0 < s < \alpha - 3/2$. Define (thanks to the classical well-posedness result) the sequence $(u_N)_{N\geq 1}$ of solutions of

$$(\partial_t^2 - \Delta)u + u^3 = 0 \tag{7}$$

with C^∞ initial data

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\,, \qquad u_1^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,.$$

The sequence $(u_N)_{N\geq 1}$ converges almost surely as $N \to \infty$ in $C(\mathbb{R}; H^s(\mathbb{T}^3))$ to a (unique) limit u which satisfies (7) in the distributional sense.

• **The type of the approximation of the initial data is crucial when we prove probabilistic well-posedness.**
• Even if we consider the approximation of the initial data by Fourier truncation there is dense set of pathological data such that the statement of Theorem 10 does not hold (we discuss this in the next slide).
• We can prove uniqueness in a suitable functional framework.
• We can consider more general randomisations (this fact had an important impact in the field).

The pathological set

• The result by Burz-Tz. provides a nice dense set Σ of initial data such that for good approximations we get nice global solutions (but for bad approximations we get divergent sequences !).

• On the other hand we also have :

Theorem 11 (Sun-Tz. 2020)

Let $0 < s < \frac{1}{2}$. Then there is a dense set $S \subset H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ such that for every $(f, g) \in S$, the sequence $(u_N)_{N \geq 1}$ of (smooth) solutions of

$$(\partial_t^2 - \Delta)u + u^3 = 0,$$

with data

$$u_0(x) = \sum_{|n| \leq N} \widehat{f}(n) e^{in\cdot x}\,, \qquad u_1(x) = \sum_{|n| \leq N} \widehat{g}(n) e^{in\cdot x}$$

do not converge. More precisely, for every $T > 0$,

$$\lim_{N \to \infty} \|u_N(t)\|_{L^\infty([0,T];H^s(\mathbb{T}^3))} = +\infty\,.$$

A remark

• The pathological set S contains a dense G_δ set of $H^s \times H^{s-1}$.

• In a very nice recent work, Camps-Gassot proved the existence of a pathological set in the more involved case of NLS.

Going beyond the Burq-Tz. result

Theorem 12 (Oh-Pocovnicu-Tz. (2019))
Let $\alpha \in (\frac{5}{4}, \frac{3}{2}]$ and $s < \alpha - 3/2$. There is divergent sequence $(c_N)_{N\geq 1}$ such that if we denote by $(u_N^\omega)_{N\geq 1}$ the solution of

$$\partial_t^2 u - \Delta u - c_N u + u^3 = 0,$$

with initial data given by

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\,, \qquad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}$$

then for almost every ω there exists $T_\omega > 0$ such that $(u_N^\omega)_{N\geq 1}$ converges in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$.

- A triviality result motivating the introduction of c_N can also be obtained.

$\boxed{\text{Remarks}}$

• The result by Oh-Pocovnicu-Tz. was the first step in the study of the nonlinear wave equation in Sobolev spaces of negative indexes. It was very recently improved to $\alpha \in (1, \frac{3}{2}]$ in an impressive work by Bringmann (using techniques developed by Gubinelli et al.).

• A recent work by Bringmann-Deng-Nahmod-Yue on the Gibbs measures for 3d NLS provides a first step to the case $\alpha = 1$ (the singular part of the support of the associated Gibbs measure).

• **A problem that I am very interested in:** Can we globalize the solutions obtained in the results of Oh-Pocovnicu-Tz. and Bringmann.

$$(i\partial_t + \Delta)u - |u|^2 u = 0, \quad u(0,x) = u_0(x) \quad x \in \mathbb{T}^2. \tag{8}$$

• (8) is a Hamiltonian PDE. Therefore

$$E(u) = \int_{\mathbb{T}^2} \Big(|\nabla_x u(t,x)|^2 + |u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4 \Big) dx$$

is a (formally) conserved quantity for (8).

• The Gibbs measure associated with (8) is a **renormalisation** of the completely formal object

$$\exp(-E(u))du\,.$$

• The measure obtained by this **renormalisation** is absolutely continuous with respect to the gaussian measure induced by

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a family of independent (complex valued) gaussians from $\mathcal{N}_{\mathbb{C}}(0,1)$ (notice that this is the same object as the one we considered previously on an arbitrary manifold).

Theorem 13 (Bourgain (1996))

• *Let* $(u_N^\omega)_{N\geq 1}$ *be the sequence of solutions of*

$$(i\partial_t + \Delta)u - |u|^2 u = 0 \tag{9}$$

with C^∞ *initial data given by*

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x} .$$

For every $s < 0$*, the sequence*

$$\left(\exp\left(\frac{it}{2\pi^2} \|u_N^\omega(t)\|_{L^2}^2\right) u_N^\omega(t) \right)_{N\geq 1}$$

converges almost surely in $C(\mathbb{R}; H^s(\mathbb{T}^2))$ *to a limit which satisfies a renormalised version of* (9).

• *Moreover, the Gibbs measure is invariant under the resulting flow.*

$\boxed{\text{Remarks}}$

• The statement of the results by Bourgain and Burq-Tz. are similar. A notable difference is that in the Bourgain theorem, in order to obtain a limit one needs to reanormalise the sequence of approximate solutions $(u_N^\omega)_{N\geq 1}$. Moreover in Bourgain's theorem the randomisation is "rigid".

• We can formulate the Bourgain theorem in the spirit of the result by Oh-Pocovnicu-Tz. More precisely, one can prove the convergence of the solutions of

$$i\partial_t u + \Delta u + c_N u - |u|^2 u = 0$$

with data

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x},$$

where $(c_N(\omega))_{N\geq 1}$ is a sequence of real numbers almost surely divergent to $+\infty$.

Singular stochastic PDE's

• The set of problems considered in the previous slides is close to the analysis of parabolic PDE's in the presence of a singular random source term (noise).
• The closest to the previously considered models is the nonlinear heat equation

$$\partial_t u - \Delta u + u^3 = \xi, \quad u(0,x) = 0\,, \quad x \in \mathbb{T}^3. \tag{10}$$

• Here ξ is the space-time white noise on $[0,\infty[\times\mathbb{T}^3$. It is the source term ξ which represents the singular randomness in (10) (in the previous slides it was the low regularity random initial data which represented the singular randomness).
• The white noise on $[0,\infty[\times\mathbb{T}^3$ may be written as

$$\xi = \sum_{n\in\mathbb{Z}^3} \dot{\beta}_n(t) e^{in\cdot x}, \tag{11}$$

where β_n are independent Brownian motions, conditioned by $\beta_n = \overline{\beta_{-n}}$ (β_0 is real and for $n \neq 0$, β_n is with values in $\mathbb{C}$).

Singular stochastic PDE's (sequel)

• For $N \gg 1$, an approximation of ξ by smooth functions is given by $\xi_N(t,x) = \rho_N \star \xi$ where $\rho_N(t,x) = N^5 \rho(N^2 t, Nx)$ with ρ a test function with integral 1 on $[0,\infty[\times\mathbb{T}^3$.

Theorem 14 (Hairer (2014), Mourrat-Weber (2018))
There is a sequence $(c_N)_{N\geq 1}$ of positive numbers, divergent as $N \to \infty$ such that if we denote by u_N the solution of

$$\partial_t u_N - \Delta u_N - c_N u_N + u_N^3 = \xi_N, \quad u(0,x) = 0$$

then $(u_N)_{N\geq 1}$ converges in suitable Hölder type spaces as $N \to \infty$.

• The initial data $u(0,x)$ can be different from zero : it suffices that it belongs to a suitable functional framework.

$\boxed{\text{Remarks}}$

• The result remains true for a noise ξ defined by

$$\xi = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^3} g_{m,n}(\omega) e^{imt} e^{in\cdot x},$$

where $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^4}$ is a family of independent complex gaussians conditioned so that ξ is real values (white noise on $\mathbb{T}\times\mathbb{T}^3$).

• The two dimensional case is treated in the work by Da Prato-Debussche (2003).

• There are other parabolic PDE's for which one can obtain results in similar spirit, the most popular being probably the KPZ equation.

On the structure of the proofs

• The proofs of the previously described results follow the same scheme.

• First, we construct local in time solutions. Then we use a global information which is either an invariant measure or an energy estimate in order to get global in time solutions.

• In order to construct the solutions locally in time, we look for the solution in the form

$$u = u_1 + u_2,$$

where u_1 contains the singular part of the solution.
• Using probabilistic arguments, close to the ones in the beginning of the lectures, we prove that u_1 has properties better than the properties given by deterministic methods. All probabilistic part of the argument is in this part of the analysis.
• In the proof of the result by Burq-Tz. we use a.s. improvements of the Sobolev embedding while all the other results use products in Sobolev spaces of negative indexes.

On the structure of the proofs (sequel)

• We then solve the problem for u_2 by purely deterministic arguments. Here the nature of the equation becomes even more important. In the case of the heat equation, the basic tool is the elliptic regularity while for the other equations we exploit the time oscillations in a crucial way (these oscillations are captured by the Bourgain spaces, for instance).

• The passage from local to global solutions in the result by Bourgain uses an invariant measure as a global control on the solutions. In the result by Burq-Tz. the globalisation is done by energy estimates. It is remarkable that in the context of the nonlinear heat equation these two techniques are also used to globalise the solutions.

A further remark

• In the work by Burq-Tz. we allow more general randomisations compared to Bourgain's work. However, the proof does not say anything about the nature of the transported by the flow initial measure while in the work by Bourgain the initial gaussian measure is quasi-invariant under the flow.

• This fact motivated recent work on quasi-invariant measures for nonlinear dispersive equations. We will come back to this.

• But let us come back to the Cauchy problem

$$\begin{cases} \partial_t^2 u - \Delta u + u^3 = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3), \end{cases}$$

where $u : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and $\mathcal{H}^s(\mathbb{T}^3) = H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$.

• The initial data is chosen among the realisations of the random series

$$u_0^\omega = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x} \qquad \text{and} \qquad u_1^\omega = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x},$$

where the scalar random variables g_n and h_n satisfy suitable assumptions.

• Recall that μ_α is the non degenerate measure on $\mathcal{H}^s(\mathbb{T}^3)$, $0 < s < \alpha - 3/2$, induced by the map

$$\omega \longmapsto (u_0^\omega, u_1^\omega)\,.$$

Theorem 15 (Burq-Tz. (2010))
Let $\alpha > \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. Let $\{u_N\}_{N\in\mathbb{N}}$ be a sequence of the smooth global solutions to

$$\partial_t^2 u - \Delta u + u^3 = 0 \tag{12}$$

with the following random C^∞-initial data:

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x} \qquad \text{and} \qquad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}.$$

Then, as $N \to \infty$, u_N converges almost surely to a (unique) limit u in $C(\mathbb{R}; H^s(\mathbb{T}^3))$, satisfying (12) in a distributional sense.

Another formulation

Theorem 16 (existence and uniqueness)
Let $\alpha > \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. There exists a full μ_α measure set $\Sigma \subset \mathcal{H}^s(\mathbb{T}^3)$ such that for every $(u_0, u_1) \in \Sigma$, there exists a unique global solution u of the non linear wave equation

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad (u(0), \partial_t u(0)) = (u_0, u_1)$$

satisfying

$$(u(t), \partial_t u(t)) \in \Big(S(t)(u_0, u_1), \partial_t S(t)(u_0, u_1)\Big) + C(\mathbb{R}_t; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)).$$

Furthermore, if we denote by

$$\Phi(t)(u_0, u_1) \equiv (u(t), \partial_t u(t))$$

the flow thus defined, the set Σ is invariant by the map $\Phi(t)$, namely

$$\Phi(t)(\Sigma) = \Sigma, \qquad \forall\, t \in \mathbb{R}.$$

Theorem 17 (Oh-Pocovnicu-Tz. (2018))
Let $\frac{5}{4} < \alpha \leq \frac{3}{2}$ and $s < \alpha - \frac{3}{2}$. There exists a divergent sequence $\{\beta_N\}_{N\in\mathbb{N}}$ of positive numbers such that the following holds true. There exist small $T_0 > 0$ and positive constants C, c, κ such that for every $T \in (0, T_0]$, there exists a set Ω_T of complemental probability smaller than $C\exp(-c/T^\kappa)$ such that if we denote by $\{u_N\}_{N\in\mathbb{N}}$ the smooth global solutions to

$$\begin{cases} \partial_t^2 u_N - \Delta u_N + u_N^3 - \beta_N u_N = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (u_{0,N}^\omega, u_{1,N}^\omega), \end{cases}$$

where the random initial data $(u_{0,N}^\omega, u_{1,N}^\omega)$ is given by the truncated Fourier series

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\ , \qquad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}$$

then for every $\omega \in \Omega_T$, the sequence $\{u_N\}_{N\in\mathbb{N}}$ converges to some (unique) limiting distribution u in $C([-T,T]; H^s(\mathbb{T}^3))$ as $N \to \infty$.

Probabilistic Strichartz estimates

Theorem 18

Let $\alpha > 3/2$. For every $T > 0$ and $p_1 \in [1, \infty)$, $p_2 \in [2, \infty]$,

$$\|S(t)(u_0, u_1)\|_{L^{p_1}([0,T];L^{p_2}(\mathbb{T}^3))} < \infty, \quad \mu_\alpha - \text{almost surely.}$$

• Here $S(t)(u_0, u_1)$ denotes the solution of the linear wave equation with data (u_0, u_1).

• Recall that μ_α is a non degenerate measure on $\mathcal{H}^s(\mathbb{T}^3)$, $0 < s < \alpha - 3/2$, where α measures the decay of the Fourier coefficients of the random series defining the measure μ_α.

• The proof of Theorem 18 is in the spirit of the almost sure improvement of the Sobolev embedding we already discussed.

The Picard iteration scheme

• For small times depending on (u_0, u_1), we can hope to represent the solution of the nonlinear wave equation as

$$u = \sum_{j=1}^{\infty} Q_j(u_0, u_1),$$

where Q_j is homogeneous of order j in (u_0, u_1). We have

$$\begin{aligned} Q_1(u_0, u_1) &= S(t)(u_0, u_1), \\ Q_2(u_0, u_1) &= 0, \\ Q_3(u_0, u_1) &= -\int_0^t \frac{\sin((t-\tau)\sqrt{-\Delta})}{\sqrt{-\Delta}} \Big(S(\tau)(u_0, u_1)\Big)^3 d\tau, \end{aligned}$$

etc. We have that μ_α a.s. $Q_1 \notin H^\sigma$ for $\sigma \geq \alpha - \frac{3}{2}$. However, using the probabilistic Strichartz estimates, we have that for $T > 0$,

$$\|Q_3(u_0, u_1)\|_{L_T^\infty H^1(\mathbb{T}^3)} \lesssim \|S(t)(u_0, u_1)\|^3_{L_T^3 L^6(\mathbb{T}^3)} < \infty, \quad \mu_\alpha - \text{almost surely.}$$

• Therefore the second non trivial term in the formal expansion defining the solution is μ_α a.s. more regular than the initial data !

The strategy

• The strategy will therefore be to write the solution of the nonlinear wave equation as

$$u = Q_1(u_0, u_1) + v,$$

where $v \in H^1$ and solve the equation for v by a deterministic method, exploiting the typical properties of $Q_1(u_0, u_1)$

• In the case of the cubic nonlinearity the deterministic analysis used to solve the equation for v is particularly simple, it is in fact very close to the analysis in the proof of the classical $\mathcal{H}^1$ well-posedness result. For more complicated problems the analysis of the equation for v could involve more advanced deterministic arguments.

Proposition 19 (Local well-posedness)
Consider the problem

$$(\partial_t^2 - \Delta)u + (f+u)^3 = 0\,. \tag{13}$$

There exists a constant C *such that for every time interval* $I = [a,b]$ *of size* 1*, every* $\Lambda \geq 1$*, every*

$$(u_0, u_1, f) \in H^1 \times L^2 \times L^3(I, L^6)$$

satisfying

$$\|u_0\|_{H^1} + \|u_1\|_{L^2} + \|f\|^3_{L^3(I,L^6)} \leq \Lambda$$

there exists a unique solution on the time interval $[a, a + C^{-1}\Lambda^{-2}]$ *of* (13) *with initial data*

$$u(a,x) = u_0(x), \quad \partial_t u(a,x) = u_1(x)\,.$$

• The proof is very similar to the proof of the classical local well-posedness result.

Proof of the existence and uniqueness result (sequel)

• We search the solution u under the form

$$u(t) = S(t)(u_0, u_1) + v(t)$$

• Then v solves

$$(\partial_t^2 - \Delta)v + (S(t)(u_0, u_1) + v)^3 = 0, \quad v\,|_{t=0} = 0, \quad \partial_t v\,|_{t=0} = 0. \tag{14}$$

• Thanks to the probabilistic Strichartz estimates, we have that μ_α-almost surely,

$$\begin{aligned} g(t) &= \|S(t)(u_0, u_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t), \\ f(t) &= \|S(t)(u_0, u_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\text{loc}}(\mathbb{R}_t). \end{aligned} \tag{15}$$

• The local existence for (14) follows from the local existence result of the previous slide and the first estimate in (15).

• We also deduce from the local existence result that as long as the $H^1 \times L^2$ norm of $(v, \partial_t v)$ remains bounded, the solution v of (14) exists.

Proof of the existence and uniqueness result (sequel)

- Set

$$\mathcal{E}(v(t)) = \frac{1}{2}\int_{\mathbb{T}^3}\Big((\partial_t v)^2 + |\nabla_x v|^2 + \frac{1}{2}v^4\Big)dx\,.$$

Using the equation solved by v, we now compute

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(v(t)) &= \int_{\mathbb{T}^3}\Big(\partial_t v\partial_t^2 v + \nabla_x\partial_t v\cdot\nabla_x v + \partial_t v\, v^3\Big)dx \\
&= \int_{\mathbb{T}^3}\partial_t v\Big(\partial_t^2 v - \Delta v + v^3\Big)dx \\
&= \int_{\mathbb{T}^3}\partial_t v\Big(v^3 - (S(t)(u_0,u_1)+v)^3\Big)dx.
\end{aligned}$$

- Now, we use the Cauchy-Schwarz inequality.

Proof of the existence and uniqueness result (sequel)

- Now, using the Cauchy-Schwarz inequality, we write

$$\frac{d}{dt}\mathcal{E}(v(t)) \leq C\Big(\mathcal{E}(v(t))\Big)^{1/2}\|v^3 - (S(t)(v_0, v_1) + v)^3\|_{L^2(\mathbb{T}^3)}\,.$$

- Using the Hölder inequality, we can estimate right hand-side by

$$C\Big(\mathcal{E}(v(t))\Big)^{1/2}\Big(\|S(t)(u_0, u_1)\|^3_{L^6(\mathbb{T}^3)} + \|S(t)(u_0, u_1)\|_{L^\infty(\mathbb{T}^3)}\|v^2\|_{L^2(\mathbb{T}^3)}\Big)$$

- But

$$\|v^2\|_{L^2(\mathbb{T}^3)} \leq C\Big(\mathcal{E}(v(t))\Big)^{1/2}$$

and consequently, we get the key bound

$$\frac{d}{dt}\mathcal{E}(v(t)) \leq C\Big(\mathcal{E}(v(t))\Big)^{1/2}\Big(g(t) + f(t)\Big(\mathcal{E}(v(t))\Big)^{1/2}\Big).$$

Proof of the existence and uniqueness result (sequel)

- Therefore, according to the Gronwall inequality and

$$g(t) = \|S(t)(u_0, u_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \quad \mu_\alpha - \text{ a.s.}$$
$$f(t) = \|S(t)(u_0, u_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \quad \mu_\alpha - \text{ a.s.}$$

we deduce that v exists globally in time.

- This completes the proof of the existence and the uniqueness.

The invariant set

- Define the sets

$$\Theta \equiv \Big\{ (u_0, u_1) \in \mathcal{H}^s \,:\, \|S(t)(u_0,u_1)\|^3_{L^6(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t), \\ \|S(t)(u_0,u_1)\|_{L^\infty(\mathbb{T}^3)} \in L^1_{\mathrm{loc}}(\mathbb{R}_t) \Big\}$$

and

$$\Sigma \equiv \Theta + \mathcal{H}^1.$$

- Then Σ is of full μ_α measure since so is Θ.
- The set Σ is invariant under the dynamics.
- We next turn to the result with Oh-Pocovnicu.

$$\boxed{\text{Deterministic well-posedness in } \mathcal{H}^s(\mathbb{T}^3),\ s \geq 1/2}$$

- The main tool are the Strichartz estimates that we discuss below.
- Let $\mathbb{L} = \partial_t^2 - \Delta + 1$. We write

$$\mathbb{L}^{-1} = (\partial_t^2 - \Delta + 1)^{-1}$$

to denote the Duhamel integral operator

$$\mathbb{L}^{-1}F(t) := \int_0^t \frac{\sin((t-t')\langle D\rangle)}{\langle D\rangle} F(t')dt', \quad \langle D\rangle = \sqrt{-\Delta + 1}.$$

- In other words, $u := \mathbb{L}^{-1}(F)$ is the solution to

$$\mathbb{L}u = F, \quad (u, \partial_t u)|_{t=0} = (0,0).$$

The Strichartz estimates

• The most basic regularity property of $\mathbb{L}^{-1}$ is the "wave regularity" estimate:

$$\|\mathbb{L}^{-1}(F)\|_{L^\infty([-T,T];H^s(\mathbb{T}^3))} \lesssim \|F\|_{L^1([-T,T];H^{s-1}(\mathbb{T}^3))}. \qquad (16)$$

• The Strichartz estimates are important extensions of (16).

Theorem 20

• *Let* $0 < T \leq 1$. *Then, the following estimate holds:*

$$\|\mathbb{L}^{-1}(F)\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|\mathbb{L}^{-1}(F)\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))} \lesssim \min\Big(\|F\|_{L^1([-T,T];H^{-\frac{1}{2}}(\mathbb{T}^3))}, \|F\|_{L^{\frac{4}{3}}([-T,T]\times\mathbb{T}^3)}\Big).$$

• *As a consequence (by duality) we also have that the solution of*

$$\mathbb{L}u = 0, \quad u(0) = u_0,\ \partial_t u(0) = u_1$$

satisfies

$$\|u\|_{L^4([-T,T]\times\mathbb{T}^3)} + \|u\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))} \lesssim \Big(\|u_0\|_{H^{\frac{1}{2}}} + \|u_1\|_{H^{-\frac{1}{2}}}\Big).$$

Closing the circle with a cubic nonlinearity in $\mathcal{H}^{1/2}$

• For $T > 0$, we denote by X_T the closed subspace of $C([-T,T]; H^{\frac{1}{2}}(\mathbb{T}^3))$ endowed with the norm

$$\|u\|_{X_T} = \|u\|_{L^\infty([-T,T];H^{\frac{1}{2}}(\mathbb{T}^3))} + \|u\|_{L^4([-T,T]\times\mathbb{T}^3)}\,.$$

• Then the Strichartz estimates imply that

$$\|\mathbb{L}^{-1}(u^3)\|_{X_T} \leq C\|u\|_{X_T}^3$$

• By a fixed point argument in X_T, the last estimate together with the corresponding estimate for the free evolution can be easily transformed into small data local well-posedness in $\mathcal{H}^{1/2}(\mathbb{T}^3)$ of

$$\mathbb{L}u + u^3 = 0\,.$$

• A very small variation yields large data local well-posedness in $\mathcal{H}^s(\mathbb{T}^3)$, $s > 1/2$ (we need to lose a bit of regularity in order to remove the smallness condition).

The free random evolution

- Fix $\alpha \leq \frac{3}{2}$. Recall that

$$\mathbb{L} = \partial_t^2 - \Delta + 1\,.$$

- Denote by $z_{1,N} = z_{1,N}(t,x,\omega)$ the solution to

$$\mathbb{L} z_{1,N}(t,x,\omega) = 0$$

with the random initial data $(u^{\omega}_{0,N}, u^{\omega}_{1,N})$ given by the truncated Fourier series

$$u^{\omega}_{0,N}(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^{\alpha}} e^{in\cdot x}\,, \qquad u^{\omega}_{1,N}(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,.$$

The free random evolution (sequel)

- Given $t \in \mathbb{R}$, define $g_n^t(\omega)$ by

$$g_n^t(\omega) := \cos(t\langle n\rangle)\, g_n(\omega) + \sin(t\langle n\rangle)\, h_n(\omega). \tag{17}$$

- Then, we have

$$\begin{aligned} z_{1,N}(t,x,\omega) &= \cos(t\langle\nabla\rangle)\Big(z_{1,N}(0,x,\omega)\Big) + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle}\Big(\partial_t z_{1,N}(0,x,\omega)\Big) \\ &= \sum_{|n|\leq N} \frac{g_n^t(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}. \end{aligned}$$

- Using the definitions of the Gaussian random variables $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$, we see that $\{g_n^t\}_{n\in\mathbb{Z}^3}$ defined in (17) forms a family of independent standard complex-valued Gaussian random variables conditioned that $g_n^t = \overline{g_{-n}^t}$ (in particular, g_0^t is real-valued).

The free random evolution (sequel)

• In the following, we discuss spatial regularities of various stochastic terms for fixed $t \in \mathbb{R}$. For simplicity of notation, we suppress the t-dependence and discuss spatial regularities.
• It is easy to see that $z_{1,N}$ converges almost surely to some limit z_1 in $H^{s_1}(\mathbb{T}^3)$ as $N \to \infty$, provided that

$$s_1 < \alpha - \frac{3}{2}.$$

• In particular, when $\alpha \leq \frac{3}{2}$, $z_{1,N}$ has negative Sobolev regularity (in the limiting sense) and thus $(z_{1,N})^2$ and $(z_{1,N})^3$ do not have well defined limits (in any topology) as $N \to \infty$ since it involves products of two distributions of negative regularities.

Giving a sense of the square and the cube in negative Sobolev spaces

• Let u_N be the solution to the renormalized nonlinear wave equation

$$\partial_t^2 u_N - \Delta u_N + u_N^3 - \beta_N u_N = 0$$

with the same truncated random initial data $(u^\omega_{0,N}, u^\omega_{1,N})$. By writing u as

$$u_N = z_{1,N} + v_N,$$

we see that the residual term $v_N = u_N - z_{1,N}$ satisfies the following equation:

$$\mathbb{L} v_N + v_N^3 + 3 z_{1,N} v_N^2 + 3((z_{1,N})^2 - \sigma_N) v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0$$

with zero initial data, where the parameter σ_N is defined by

$$\sigma_N := \frac{\beta_N + 1}{3}.$$

Giving a sense of the square and the cube in negative Sobolev spaces

• The key point is that the terms

$$Z_{2,N} := (z_{1,N})^2 - \sigma_N \qquad \text{and} \qquad Z_{3,N} := (z_{1,N})^3 - 3\sigma_N z_{1,N}$$

are "renormalizations" of $(z_{1,N})^2$ and $(z_{1,N})^3$.

• Here, by "renormalizations", we mean that by choosing a suitable renormalization constant σ_N, the terms $Z_{2,N}$ and $Z_{3,N}$ converge almost surely in suitable negative Sobolev spaces as $N \to \infty$.

• The regularity $s_1 < \alpha - \frac{3}{2}$ of $z_{1,N}$ (in the limit) and a basic computation (as in the first lecture) show that if the expressions $Z_{2,N} = (z_{1,N})^2 - \sigma_N$ and $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ have any well defined limits as $N \to \infty$, then their regularities in the limit are expected to be

$$s_2 < 2\Big(\alpha - \frac{3}{2}\Big) \qquad \text{and} \qquad s_3 < 3\Big(\alpha - \frac{3}{2}\Big),$$

respectively.

$\boxed{\text{Giving a sense of the square and the cube in negative Sobolev spaces}}$

• In fact, by choosing the renormalization constant σ_N as

$$\sigma_N := \mathbb{E}\Big[\Big(z_{1,N}(t,x,\omega)\Big)^2\Big],$$

we show that $Z_{j,N}$, $j = 2, 3$ converge in $H^{s_j}(\mathbb{T}^3)$ almost surely.
• The renormalization constant σ_N a priori depends on t, x but it turns out to be independent of t and x. This fact can be seen by a direct computation. It also follows from the stationarity (in both t and x) of the stochastic process $\{z_{1,N}(t,x)\}_{(t,x)\in\mathbb{R}\times\mathbb{T}^3}$.
• We will also see that, for $N \gg 1$, σ_N behaves like $\sim N^{3-2\alpha}$ when $\alpha < \frac{3}{2}$.

The first Picard iteration ansatz

- We know that the deterministic Cauchy problem for

$$\mathbb{L}v + v^3 = 0$$

is locally well-posed in $\mathcal{H}^s(\mathbb{T}^3)$ for $s \geq \frac{1}{2}$.

- Therefore, we may hope to solve the equation

$$\mathbb{L}v_N + v_N^3 + 3z_{1,N}v_N^2 + 3((z_{1,N})^2 - \sigma_N)v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0$$

uniformly in $N \in \mathbb{N}$ if we can ensure that the solution v_N to the following linear problem:

$$\mathbb{L}v_N + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0 \tag{18}$$

with the zero initial data $(v_N, \partial_t v_N)|_{t=0} = (0,0)$ remains bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ as $N \to \infty$.

- Using the wave-regularity, we see that the solution to (18) is almost surely bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ uniformly in $N \in \mathbb{N}$, provided

$$3\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \Longrightarrow \quad \alpha > \frac{4}{3}.$$

Therefore, $\alpha = \frac{4}{3}$ is the limit of the first Picard iteration ansatz.

The second Picard iteration ansatz

• In order to go below the $\alpha = \frac{4}{3}$ threshold, a new argument is needed. More precisely, we further decompose v_N as

$$v_N = z_{2,N} + w_N,$$

where $z_{2,N}$ is the solution to the following equation:

$$\begin{cases} \mathbb{L} z_{2,N} + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0 \\ (z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0,0). \end{cases}$$

• Thanks to the one degree of smoothing, we see that $z_{2,N}$ converges to some limit in $H^s(\mathbb{T}^3)$, provided that

$$s = s_3 + 1 < 3\Big(\alpha - \frac{3}{2}\Big) + 1$$

The second Picard iteration ansatz (sequel)

• In terms of the original solution u_N we have

$$u_N = z_{1,N} + z_{2,N} + w_N.$$

Note that $z_{1,N} + z_{2,N}$ corresponds to the Picard second iterate for the truncated renormalized equation.

• The equation for w_N can now be written as

$$\mathbb{L}w_N + (w_N + z_{2,N})^3 + 3z_{1,N}(w_N + z_{2,N})^2 + 3((z_{1,N})^2 - \sigma_N)(w_N + z_{2,N}) = 0,$$

with zero initial data.

• By using the second order expansion, we have eliminated the most singular term $Z_{3,N} = (z_{1,N})^3 - 3\sigma_N z_{1,N}$ in the equation for v_N.

• In the equation for w_N, there are several source terms (namely, purely stochastic terms independent of the unknown w_N) and they are precisely the quintic, septic, and nonic (i.e. degree nine) terms added in considering the Picard third iterate.

The second Picard iteration ansatz (sequel)

• It turns out that the most singular source term is the following quintic term:

$$Z_{5,N} := 3((z_{1,N})^2 - \sigma_N) z_{2,N},$$

where we recall that $z_{2,N}$ is the solution to

$$\begin{cases} \mathbb{L} z_{2,N} + ((z_{1,N})^3 - 3\sigma_N z_{1,N}) = 0, \\ (z_{2,N}, \partial_t z_{2,N})|_{t=0} = (0,0). \end{cases}$$

• As we already mentioned, the term $Z_{2,N} = (z_{1,N})^2 - \sigma_N$ and the second order term $z_{2,N}$ pass to the limits in $H^s(\mathbb{T}^3)$ for $s < 2(\alpha - \frac{3}{2})$ and $s < 3(\alpha - \frac{3}{2}) + 1$, respectively.
• In order to make sense of the product of $Z_{2,N}$ and $z_{2,N}$ by deterministic paradifferential calculus, we need the sum of the two regularities to be positive, namely

$$2\left(\alpha - \frac{3}{2}\right) + 3\left(\alpha - \frac{3}{2}\right) + 1 > 0 \quad \Longrightarrow \quad \alpha > \frac{13}{10}.$$

The second Picard iteration ansatz (sequel)

• Otherwise, i.e. for $\alpha \le \frac{13}{10}$, we will need to make sense of the product of $Z_{2,N}$ and $z_{2,N}$ using stochastic analysis.
• In either case, when the second term in

$$Z_{5,N} := 3((z_{1,N})^2 - \sigma_N)z_{2,N},$$

has positive regularity $3(\alpha - \frac{3}{2}) + 1 > 0$, i.e. $\alpha > \frac{7}{6}$, we show that the product

$$3((z_{1,N})^2 - \sigma_N)z_{2,N}$$

(in the limit) inherits the regularity from $Z_{2,N} = (z_{1,N})^2 - \sigma_N$, allowing us to pass to a limit in $H^s(\mathbb{T}^3)$ for

$$s < 2\left(\alpha - \frac{3}{2}\right).$$

The equation for w_N

• Once we are able to pass the term

$$Z_{5,N} = 3((z_{1,N})^2 - \sigma_N)z_{2,N},$$

in the limit $N \to \infty$, the main issue in solving the equation for w_N by the deterministic Strichartz theory is to ensure that the solution of

$$\mathbb{L}w + 3((z_{1,N})^2 - \sigma_N)z_{2,N} = 0 \tag{19}$$

with zero initial data remains bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$ as $N \to \infty$.

• Using again one degree of smoothing under the wave Duhamel operator, we see that the solution to (19) is almost surely bounded in $H^{\frac{1}{2}}(\mathbb{T}^3)$, provided

$$2\left(\alpha - \frac{3}{2}\right) + 1 > \frac{1}{2} \quad \Longrightarrow \quad \alpha > \frac{5}{4}.$$

• This explains the restriction $\alpha > \frac{5}{4}$ in our result.

The limit equation

• In the proof of our result, we apply the deterministic Strichartz theory and show that w_N converges almost surely to some limit w.

• Along with the almost sure convergence of $z_{1,N}$ and $z_{2,N}$ to some limits z_1 and z_2, respectively, we conclude from the decomposition

$$u_N = z_{1,N} + z_{2,N} + w_N$$

that u_N converges almost surely to

$$u := z_1 + z_2 + w\,.$$

• By taking a limit in the equation for w_N, we see that w is almost surely the solution to

$$\begin{cases} \mathbb{L}w + (w + z_2)^3 + 3z_1(w + z_2)^2 + 3Z_2 w + 3Z_5 = 0 \\ (w, \partial_t w)|_{t=0} = (0, 0), \end{cases}$$

where Z_2 and Z_5 are the limits of $Z_{2,N}$ and $Z_{5,N}$, respectively.

• This equation for w may be seen as the limit equation for $u - z_1 - z_2$.

On the Bringmann improvement

• In his work allowing to treat $\alpha > 1$ Bringmann uses a finer (the so called para-control) ansatz by writing

$$u_N = z_{1,N} + z_{2,N} + R_N + w_N,$$

where R_N has the same regularity as $z_{2,N}$ and some high frequency structure (as I explained on the blackboard in the previous lecture).
• More importantly, it exploits multi-linear smoothing effects in the stochastic objects.
• Oh-Wang-Zine succeeded to get the $\alpha > 1$ result without using the para-control ansatz.

• For $\alpha > 1$ the estimates on the random objects become thus much more involved but one still uses the same basic probabilistic effects as in the beginning of the lectures.

Quasi-invariant measures

- Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be a torus of dimension d.
- If $f : \mathbb{T}^d \to \mathbb{C}$ is a C^∞ function then for every $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

where

$$\widehat{f}(n) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx$$

then

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\widehat{f}(n)|^2 \,.$$

- The norm H^s is induced from a natural scalar product

$$(f,g)_s = \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} \widehat{f}(n)\overline{\widehat{g}(n)}$$

which makes $H^s(\mathbb{T}^d)$ a Hilbert space.

The gaussian measure μ_s

- We wish to define a gaussian measure of the form

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as a measure on a suitable functional space.

- Formally

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du = Z^{-1} \exp\Big(-\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2s} |\widehat{u}(n)|^2\Big) \prod_{n\in\mathbb{Z}^d} d\,\widehat{u}(n)$$

and the last expression makes think about the well defined object

$$\prod_{n\in\mathbb{Z}} Z_n^{-1} \exp\Big(-\langle n\rangle^{2s} |\widehat{u}(n)|^2\Big) d\,\widehat{u}(n),$$

where we formally wrote

$$Z^{-1} = \prod_{n\in\mathbb{Z}} {Z_n}^{-1}\,.$$

$\boxed{\text{The gaussian measure } \mu_s \text{ (sequel)}}$

- Therefore, we can define the measure μ_s

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as the image measure by the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}^d} e^{inx}\, \frac{g_n(\omega)}{\langle n\rangle^s}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}^d}$ are i.i.d. complex gaussian random variables with mean 0 and variances 1, on a probability space $(\Omega, \mathcal{F}, p)$.

- **Question :** μ_s is a measure on which space ?

The gaussian measure μ_s (sequel)

- We can write for $N < M$

$$\Big\| \sum_{N \leq |n| \leq M} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s} \Big\|^2_{L^2(\Omega; H^\sigma(\mathbb{T}^d))} \simeq \sum_{N \leq |n| \leq M} \frac{\langle n \rangle^{2\sigma}}{\langle n \rangle^{2s}}$$

which tends to zero as $N \to \infty$, provided

$$\sigma < s - \frac{d}{2}.$$

- Therefore

$$\sum_{n \in \mathbb{Z}^d} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s} \in L^2(\Omega; H^\sigma(\mathbb{T}^d)).$$

The gaussian measure μ_s (sequel)

• We conclude that the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}^d} e^{inx} \, \frac{g_n(\omega)}{\langle n\rangle^s}$$

defines a probability measure on $H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$. In addition

$$\mu_s(H^{s-\frac{d}{2}}(\mathbb{T}^d)) = 0\,.$$

• In particular

$$\mu_s(H^s(\mathbb{T}^d)) = 0\,.$$

• In this constriction $H^s(\mathbb{T}^d)$ is canonical but $H^\sigma(\mathbb{T}^d)$ is not, it may be replaced for instance by $W^{\sigma,\infty}(\mathbb{T}^d)$.

The Cameron-Martin theorem

• **Question :** How behaves μ_s under transformations ?

Theorem 21 (Cameron-Martin 1944)
Let $f \in H^\sigma(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$ and let μ_f be the image of μ_s under the map from $H^\sigma(\mathbb{T}^d)$ to $H^\sigma(\mathbb{T}^d)$ defined by

$$u \longmapsto f + u\,.$$

Then μ_f is absolutely continuous with respect to μ_s if and only if

$$f \in H^s(\mathbb{T}^d).$$

• Recalling that formally

$$d\mu_s(u) = Z^{-1}\, e^{-\|u\|^2_{H^s}}\, du$$

we may expect that

$$\frac{d\mu_f(u)}{d\mu_s(u)} = e^{-\|f\|^2_{H^s} - 2(u,f)_s}\,,$$

where $(\cdot,\cdot)_s$ stands for the H^s scalar product.

Proof of the Cameron-Martin theorem for μ_s

• Let $f \in H^s(\mathbb{T}^d)$. Since we expect that the Radon-Nykodim derivative is $\exp\Big(-\|f\|^2_{H^s} - 2(u,f)_s\Big)$ the whole issue is to show that $(u,f)_s < \infty$, μ_s almost surely which is equivalent to

$$\sum_{n\in\mathbb{Z}^d} \langle n\rangle^s \widehat{f}(n)\,\overline{g_n(\omega)} < \infty, \qquad \text{a.s.}$$

which directly results directly from the independence and $f \in H^s(\mathbb{T}^d)$.
• Let now $f \notin H^s(\mathbb{T}^d)$. Then there is $g \in H^s$ such that $(f,g)_s = \infty$. Consider the set

$$A = \{u \in H^\sigma \,:\, (g,u)_s < \infty\}.$$

We already checked that $\mu_s(A) = 1$ (replace f by g in the discussion of the first half of the slide). The image of A under our shift is the set B defined by

$$B = \{u + f, \quad u \in A\}.$$

Clearly $A \cap B = \emptyset$ and therefore $\mu_s(B) = 0$. Thus we found a set of measure 1 which is sent by the shilt by f map to a set of measure 0. This completes the proof.

Invariance of μ_s under the free Schrödinger evolution

Proposition 22
Let $S(t) = e^{it\Delta}$. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u \longmapsto S(t)(u)$. Then $\mu_s(t) = \mu_s$.

Proof. We have that

$$S(t)\Big(\sum_{n\in\mathbb{Z}^d} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}\Big) = \sum_{n\in\mathbb{Z}^d} e^{inx}\,\frac{e^{-itn^2}g_n(\omega)}{\langle n\rangle^s}$$

which has the same distribution as

$$\sum_{n\in\mathbb{Z}^d} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}$$

because $e^{-itn^2}g_n(\omega)$ has the same distribution as $g_n(\omega)$ (invariance of complex gaussians by rotations). This completes the proof.

$\boxed{\text{A remark}}$

• Even in $1d$, for a fixed sequence $(c_n)_{n\in\mathbb{Z}}$ the free Schrödinger evolution

$$\sum_{n\in\mathbb{Z}} c_n\, e^{inx}\, e^{-itn^2}$$

may have a complicated behaviour depending on the nature of the number t (leading to interesting number theory considerations) but the statistical behaviour under μ_s is the same for each time t.

Transport of μ_s under nonlinear transformations

Question : How behaves μ_s under the flow of the nonlinear Schrödinger equation (NLS) ? Let us start by the dispersionless model :

Theorem 23 (Oh-Sosoe-Tz. (2017))
Let $d = 1$, $s \geq 1$ be an integer and $0 < \sigma < s - 1/2$. Let $\rho_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves

$$i\partial_t u = |u|^2 u, \quad u|_{t=0} = u_0\,. \tag{20}$$

Then for $t \neq 0$, the measure $\rho_s(t)$ is not absolutely continuous with respect to μ_s.

• The solution of (20) is given by

$$u(t,x) = u_0(x)\; e^{-it|u_0(x)|^2} \tag{21}$$

and the idea behind the proof is to show that a typical regularity property of the data resulting from the iterated logarithm law associated with μ_s is destroyed by the time oscillation in formula (21).

Transport of μ_s under nonlinear transformations (sequel)

But we also have :

Theorem 24 (Deng-Sun-Tz. 2022)

Let $s > 2$ and $1 \leq \sigma < s-1$. Let $p \geq 2$ be an even integer. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T}^2)$ to $H^\sigma(\mathbb{T}^2)$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u = |u|^p u, \quad u|_{t=0} = u_0 \,. \tag{22}$$

Then $\mu_s(t)$ is absolutely continuous with respect to μ_s. In other words, μ_s is quasi-invariant under the flow of (22). In particular for fixed t, x the law of $u(t,x)$ has a density with respect to the Lebesgue measure on $\mathbb{C}$.

$\boxed{\text{Remarks}}$

• Previously, we had similar results for NLS in $1d$, for the nonlinear wave equations in dimensions ≤ 3 (with energy sub-critical nonlinearities), for the gKdV equation and for BBM type models.

• The first result for measures in negative Sobolev spaces is by Oh-Seong in the context of 4NLS.

• The $3d$ NLS does not seem out of reach ...

• Depending on the equation, we have more or less informations on the resulting Radon-Nykodim derivatives.

A corollary (L^1 stability for the corresponding Liouville equation)

Theorem 25
Let $s > 2$. Let $f_1, f_2 \in L^1(d\mu_s)$ and let $\Phi(t)$ be the flow of

$$(i\partial_t + \Delta)u = |u|^{2p}u, \quad u|_{t=0} = u_0\,,$$

defined μ_s a.s. Then for every $t \in \mathbb{R}$, the transports of the measures

$$f_1(u)d\mu_s(u), \quad f_2(u)d\mu_s(u)$$

by $\Phi(t)$ are given by

$$F_1(t,u)d\mu_s(u), \quad F_2(t,u)d\mu_s(u)$$

respectively, for suitable $F_1(t,\cdot), F_2(t,\cdot) \in L^1(d\mu_s)$. Moreover

$$\|F_1(t) - F_2(t)\|_{L^1(d\mu_s)} = \|f_1 - f_2\|_{L^1(d\mu_s)}\,.$$

• Local in time bounds for other distances are obtained in a recent work by work by Forlano-Seong. There are many further things to be understood.

$\boxed{\text{Remarks}}$

• The above results are restricted to relatively regular solutions of the equation (cf. the assumption $s > 2$) because in my present understanding the question of quasi-invariance seems *strictly more complicated* than the question of proving the existence of the dynamics (an infinite dimensional phenomenon).
• For exemple, in the context of the impressive recent results by Deng-Nahmod-Yue for NLS with low regularity gaussian data, the question of the propagation of the gaussianity by the flow of the equation seems completely open.
• A similar remark applies to the earlier probabilistic well-posedness results by Nicolas Burq and myself on the nonlinear wave equation we discussed in the previous lecture and the result by Colliander-Oh on the 1d NLS.

Remarks (sequel)

• I however expect that the methods and the ideas developed in the work on probabilistic well-posedness may become useful in quasi-invariance questions. Ideally, one day we would succeed to have a quasi-invariance result for a deterministically ill-posed posed problem.
• A very interesting recent work by Forlano-Tolomeo uses the probabilistic well-posedness in the context of a quasi-invariance problem. It is however not clear whether in their model (1d fractional NLS) the probabilistic well-posendess is really needed.
• In May 2022, Leonardo Tolomeo announced me that he was able to prove the quasi-invariance of the gaussian measures in the context the probabilistic well-posedness results by Nicolas Burq and myself discussed earlier in these lectures. I am looking very much forward to see the details of his work.

Methods

• Roughly speaking, presently, we have two different methods to prove this kind of quasi-invariance results :

• **Method 1 :** Using the *time oscillations* (dispersive estimates).

• **Method 2 :** Using the *random oscillations* (in the spirit of the analysis we did in the first lecture).

• In both methods, we do not study directly the evolution of the gaussian measure μ_s but the evolution of ρ_s defined by

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where $R_s(u)$ is a suitable correction and where χ is a continuous function with a compact support and where $H(u)$ is the Hamiltonian of the equation under consideration (conserved by the flow). We formally have

$$e^{-R_s(u)}d\mu_s(u) = Z^{-1}e^{-R_s(u)}e^{-\|u\|^2_{H^s}}du = Z^{-1}e^{-E_s(u)}du\,,$$

where

$$E_s(u) = \|u\|^2_{H^s} + R_s(u)\,.$$

Methods (sequel)

• The correction $R_s(u)$ in the energy functional

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)$$

is of fundamental importance and there are different intuitions behind its construction : normal form reductions, traces of complete integrability, modulated energies, ...

• Interestingly, in some cases the construction of $R_s(u)$ requires renormalisation arguments.

• However, an important feature is that we *do not renormalise the equation which stays always the same*. Instead, we consider renormalised functionals associated with the equation with data distributed according to a gaussian field.

On method 1

- Let $\Phi(t)$ be the flow of the PDE under consideration.
- Formally the transported measure is given by

$$Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, du = Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, e^{E_s(u)} e^{-E_s(u)} du$$

which can be interpreted as the (relatively) well defined object

$$e^{-\left(E_s(\Phi(t)(u))-E_s(u)\right)}\, \chi(H(u)) e^{-R_s(u)} d\mu_s(u)\,.$$

- Therefore we hope that the Radon-Nykodim derivative of the transport of ρ_s is given by

$$e^{-\left(E_s(\Phi(t)(u))-E_s(u)\right)}$$

- **Problem :** In $E_s(\Phi(t)(u)) - E_s(u)$ both terms are strongly diverging on the support of μ_s but the hope is to find some cancellations thanks to PDE smoothing estimates.

On method 1 (sequel)

• More precisely, one can write

$$E_s(\Phi(t)(u)) - E_s(u) = \int_0^t \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau} d\tau.$$

Set

$$G_s(\tau) = \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau}.$$

We will be done, if we can prove that

$$\Big| \int_0^t G_s(\tau) d\tau \Big| \leq C_{H(u)} \|u\|^{\theta}_{H^{s-\frac{d}{2}-}},$$

for a suitable choice of $R_s(u)$ and for a suitable number θ.

• If E_s is a conserved quantity (Gibbs measures) then $G_s = 0$ and one expects an invariant measure. However, this may not be true at the level of the approximated finite dimensional models and a serious difficulty may appear (cf. works by Nahmod-Oh-Rey Bellet-Staffilani, Tz.-Visciglia, Genovese-Luca-Valeri, ...).

On method 1 (sequel)

• If $\theta < 2$ the Randon-Nykodim density is indeed given by

$$e^{-\left(E_s(\Phi(t)(u))-E_s(u)\right)}$$

in the sense that it is the natural limit of the corresponding (perfectly well defined) finite dimensional densities.

• If $\theta \geq 2$, we can define the Radon-Nykodim density of the transport of

$$\exp\left(-\|u\|^m_{H^{s-\frac{d}{2}-}}\right)\chi(H(u))\,e^{-R_s(u)}d\mu_s(u),$$

where $m \gg 1$ (depending on θ).

• **Remark.** It would be interesting to replace

$$\left|\int_0^t G_s(\tau)d\tau\right| \leq C_{H(u)}\|u\|^\theta_{H^{s-\frac{d}{2}-}},$$

with more subtle estimates.

On method 2

- Let $A \subset H^\sigma(\mathbb{T})$ be a measurable set.
- Recall that

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where χ is a continuous function with a compact support and $H(u)$ is the Hamiltonian of the equation under consideration.

- Then

$$\frac{d}{dt}\rho_s(\Phi(t)(A))\Big|_{t=\bar{t}} = \frac{d}{dt}\rho_s(\Phi(t)(\Phi(\bar{t})(A)))\Big|_{t=0}$$

which is **formally** equal to

$$\int_{\Phi(\bar{t})(A)} \frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0}\, d\rho_s(u)$$
$$\leq \left\| \frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0} \right\|_{L^p(\rho_s)} \Big(\rho_s(\Phi(\bar{t})(A))\Big)^{1-\frac{1}{p}}$$

On method 2 (sequel)

• We would be done if we show that

$$\left\| \frac{d}{dt} E_s(\Phi(t)(A)) \Big|_{t=0} \right\|_{L^p(\rho_s)} \leq Cp, \quad p \gg 1 \,. \tag{23}$$

In the proof of the last inequality we only exploit the random oscillations of the initial data.

• Important observation : if we are only interested in the qualitative statement of quasi-invariance then in (23) we can suppose that A included in a bounded set of a Banach space $\mathcal{H}$ which is of full measure such that the PDE under consideration is globally well posed in $\mathcal{H}$ (existence, uniqueness and persistence of regularity).

• Let us **formally** show how we use (23) (similarly to the uniqueness for $2d$ Euler) to get the quasi-invariance. Set

$$x(t) = \rho_s(\Phi(t)(A)) \,.$$

Thanks to (23) we have

$$\dot{x}(t) \leq Cp(x(t))^{1-\frac{1}{p}}$$

On method 2 (sequel)

Therefore

$$\frac{d}{dt}\Big((x(t))^{\frac{1}{p}}\Big) \leq C\,.$$

• An integration yields

$$(x(t))^{\frac{1}{p}} - (x(0))^{\frac{1}{p}} \leq Ct$$

Therefore, if $x(0) = 0$ then

$$x(t) \leq (Ct)^p$$

which goes to zero as $p \to \infty$, provided $Ct < 1$.
• Since the constant C is uniform we can iterate the last argument and achieve any time.
• The above argument may become rigorous if we use some approximation arguments resulting from the Cauchy problem theory of the equation under consideration.

Final remarks

• Basically, it may look that Method 2 performs better for equations with weaker dispersion.

• I do not see yet an efficient way to combine Method 1 and Method 2 ...

• In the work on 2d NLS with Deng and Sun, we follow Method 2 with several key novelties. One of them is that thanks to the structure of the resonant set we can use a normal from reduction and then use the time oscillations via the Strichartz estimates for the linear equation (a similar idea was used in my work with Hani-Pausader-Visciglia on solutions of NLS with growing higher Sobolev norms).

• We are not able so far to use the recent refined resolution ansatz (as the random averaging operators) in the context of quasi-invariance of gaussian measures. It would be very interesting to clarify whether it may be possible. This is what I am presently trying to understand ...

2D NLS analysis, the setup

- Write
$$v(t) = e^{-it\Delta}u(t), \quad v(t) = \sum_k v_k(t)e^{ik\cdot x}.$$
- If $u(t)$ solves $i\partial_t u + \Delta u = |u|^2 u$, then
$$\partial_t v_k = \frac{1}{i}\sum_{k_1-k_2+k_3=k} e^{-it\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2}v_{k_3},$$

where

$$\Phi(\vec{k}) := |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2 = 2(k_1-k_2)\cdot(k_2-k_3).$$

- We have
$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 = -\frac{1}{4}\mathrm{Im}\sum_{\substack{k_1-k_2+k_3-k_4=0\\ k_2\neq k_1,k_3}} \psi_{2s}(\vec{k})e^{-it\Phi(\vec{k})}v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4},$$

$$\psi_{2s}(\vec{k}) = |k_1|^{2s} - |k_2|^{2s} + |k_3|^{2s} - |k_4|^{2s}.$$

• Set

$$\mathcal{N}_{0,t}(v) = \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \psi_{2s}(\vec{k}) \frac{e^{-it\Phi(\vec{k})}}{-i\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4},$$

$$\mathcal{R}_{0,t}(v) = \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) v_{k_1}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4}$$

$$\mathcal{R}_{1,1,t}(v) = 2 \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it\Phi(\vec{k})} \sum_{p_1-p_2+p_3=k_1} e^{-it\Phi(\vec{p})} v_{p_1}\overline{v}_{p_2} v_{p_3}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4},$$

$$\mathcal{R}_{1,2,t}(v) = -2 \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it\Phi(\vec{k})} \sum_{q_1-q_2+q_3=k_2} e^{it\Phi(\vec{q})} v_{k_1}\overline{v}_{q_1} v_{q_2}\overline{v}_{q_3} v_{k_3}\overline{v}_{k_4}.$$

• Defining

$$E_{s,t}(v) := \frac{1}{2}\|v\|_{H^s}^2 + \frac{1}{4}\mathrm{Im}\mathcal{N}_{0,t}(v)$$

we obtain that along the NLS flow, we have

$$\frac{d}{dt}E_{s,t}(v) := \frac{1}{4}\mathrm{Im}\Big[\mathcal{R}_{1,1,t}(v) + \mathcal{R}_{1,2,t}(v) - \mathcal{R}_{0,t}(v)\Big]$$

• Let us look at the simplest (resonant) term

$$\mathcal{R}_{0,t}(v) := \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) v_{k_1}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4}.$$

• W.L.O.G., we assume that $v_{k_j} = \widehat{P_{N_j}v}(k_j)$ and $N_{(1)} \geq N_{(2)} \geq N_{(3)} \geq N_{(4)}$ are the rearrangement of N_1, N_2, N_3, N_4.

• We have $|\psi_{2s}(\vec{k})| \lesssim N_{(1)}^{2s-2} N_{(3)}^2$ and therefore

$$|\mathcal{R}_{0,t}(v)| \lesssim N_{(1)}^{2s-2} N_{(3)}^2 \int_0^{2\pi} \int_{\mathbb{T}^2} e^{it\Delta} f_1 \cdot \overline{e^{it\Delta} f_2} e^{it\Delta} f_3 \cdot \overline{e^{it\Delta} f_4} dt dx,$$

where $\widehat{f}_j(k_j) = |v_{k_j}|$. The space-time integral can be treated using the bilinear Strichartz estimate. Due to the unavoidable loss $N_{(3)}^{0+}$, we have

$$|\mathcal{R}_{0,t}(v)| \lesssim \|\mathbf{P}_{N_{(1)}} v\|_{H^{s-1}} \|\mathbf{P}_{N_{(2)}} v\|_{H^{s-1}} \|\mathbf{P}_{N_{(3)}} v\|_{H^{2+}} \|\mathbf{P}_{N_{(4)}} v\|_{L^2}.$$

• No matter how large s is, the above estimate is not enough for our need, as $v \in H^{(s-1)-}$ almost surely. Nevertheless, we are ϵ-close to what we expect (for s large).

Exploiting the random oscillation

• By Method II, what we are allowed reduce the estimate to $t = 0$ and average on the support of the measure. So we have access to the probability toolbox: Wiener chaos estimate: $l-$linear Gaussian sum:

$$\mathcal{T}_l := \sum_{k_1,\cdots,k_l} c_{k_1,\cdots,k_l} g_{k_1}(\omega)\cdots g_{k_l}(\omega),$$

for any $p \geq 2$, $\|\mathcal{T}_l\|_{L^p_\omega} \leq C p^{\frac{l}{2}} \|\mathcal{T}_l\|_{L^2_\omega}$.

• The pairing contributions $(k_1 = k_2, k_3 = k_4), (k_1 = k_4, k_2 = k_3)$ in $\mathcal{R}_{0,t}(v)$ disappear by taking the imaginary part, it is reduced to estimate

$$p^2 \Big\| \sum_{\substack{k_1-k_2+k_3-k_4=0,\\ k_2\neq k_1,k_3\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) \mathbf{1}_{|k_j|\sim N_j} \frac{g_{k_1}(\omega)\overline{g}_{k_2}(\omega)g_{k_3}(\omega)\overline{g}_{k_4}(\omega)}{\langle k_1\rangle^s \langle k_2\rangle^s \langle k_3\rangle^s \langle k_4\rangle^s} \Big\|_{L^2_\omega}$$

Consider the worst case, say $N_1 \sim N_2 \gg N_3 + N_4 = O(1)$, the above quantity can be crudely bounded by $p^2 N_{(1)}^{2s-2} \cdot N_{(1)}^{-2s+1} = p^2 N_{(1)}^{-1}$. By interpolating with the deterministic bound in the last slide, we conclude that $\|\mathrm{Im}\mathcal{R}_{0,t}(v)|_{t=0}\|_{L^p_\omega} \leq Cp$.

The key cancellation

- The treatment for $\mathcal{N}_{0,t}(v)$ follows from the similar analysis + resonance decomposition according to the value of $\Phi(\vec{k})$.

- However, the estimate for the second generations $\mathcal{R}_{1,j,t}(v), j = 1, 2$ requires another algebraic cancellation.

- The reason is that in the high-high-low-low-low-low regime, the most problematic contribution is the paring of two dominant frequencies living in different generations. These types of pairing prevent us to gain from the Wiener chaos.

The key cancellation (sequel)

- Written in formula, these two pairing configurations are:

$$\mathcal{S}_{1,1,1}(v) :=$$

$$4\sum_{k_1\neq k_2}|v_{k_2}|^2 \sum_{\substack{|k_3|+|k_4\ll|k_1|,|k_2| \\ |p_2|+|p_3|\ll|k_1|,|k_2| \\ k_3-k_4=k_2-k_1 \\ p_2-p_3=k_2-k_1}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it(|k_3|^2-|k_4|^2+|p_2|^2-|p_3|^2)} v_{k_3}\overline{v}_{k_4}\overline{v}_{p_2}v_{p_3},$$

and

$$\mathcal{S}_{1,1,2}(v) :=$$

$$4\sum_{k_1,k_3}|v_{k_3}|^2 \sum_{\substack{|k_2|+|k_4|\ll|k_1|,|k_3| \\ |p_1|+|p_3|\ll|k_1|,|k_3| \\ p_1+p_3=k_1+k_3 \\ k_2+k_4=k_1+k_3}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{it(|k_2|^2+|k_4|^2-|p_1|^2-|p_3|^2)} \overline{v}_{k_2}\overline{v}_{k_4}v_{p_1}v_{p_3}.$$

$\boxed{\text{The key cancellation (sequel)}}$

• To understand the hidden cancellation, for $\mathcal{S}_{1,1,1}(v)$, one can think about the sum is taken over $|k_3|, |k_4|, |p_2|, |p_3| = O(1)$, then

$$\frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} \approx \frac{|k_1|^{2s} - |k_2|^{2s}}{|k_1|^2 - |k_2|^2},$$

then the second sum in the definition of $\mathcal{S}_{1,1,1}$ is completely decoupled as $|\cdots|^2$ and we deduce that $\mathcal{S}_{1,1,1}$ is almost real.

Thank you for your attention !