October 2014, Orsay

The NLS on product spaces and applications

Nikolay Tzvetkov

Cergy-Pontoise University

based on joint work with

Zaher Hani, Benoit Pausader and Nicola Visciglia

A basic result

- Consider the Cauchy problem associated with the defocusing NLS

$$(i\partial_t + \Delta - |U|^2)U = 0, \quad U|_{t=0} = U_0(x), \quad t \in \mathbb{R},\ x \in \mathbb{R} \times \mathbb{T}^2. \qquad (1)$$

- The following quantity (energy) is conserved by the flow of (1)

$$E(U) = \|U\|^2_{H^1(\mathbb{R}\times\mathbb{T}^2)} + \frac{1}{2}\|U\|^4_{L^4(\mathbb{R}\times\mathbb{T}^2)} .$$

Therefore there is a control on the H^1 norm of the solutions of (1).

Proposition 1 (follows from work by Bourgain)
Let $s \geq 1$. For every $U_0 \in H^s(\mathbb{R} \times \mathbb{T}^2)$ there is a unique solution of (1) in $C(\mathbb{R}; H^s(\mathbb{R} \times \mathbb{T}^2))$. In addition there is a positive constant A (independent of s, t and U_0) such that

$$\|U(t)\|_{H^s(\mathbb{R}\times\mathbb{T}^2)} \lesssim (1 + |t|)^{A(s-1)} .$$

(the implicit constant is independent of t but depends on U_0 and s).

Solutions with unbounded higher Sobolev norms

• **Problem** (Bourgain, GAFA'2000 special volume) : Can we find, for some $s > 1$, an H^s global solution of the cubic defocusing NLS such that the H^s norm of this solution **does not remain bounded** as t goes to infinity ? If yes can we quantify the growth ?

Theorem 2 (HPTV'2013)

Let us fix $s \geq 30$ and $\varepsilon > 0$. Then there exists $U_0 \in H^s(\mathbb{R} \times \mathbb{T}^2)$ such that $\|U_0\|_{H^s} < \varepsilon$ and such that the corresponding solution of

$$(i\partial_t + \Delta - |U|^2)U = 0, \quad U|_{t=0} = U_0(x), \quad t \in \mathbb{R}, \; x \in \mathbb{R} \times \mathbb{T}^2.$$

satisfies

$$\limsup_{t \to \infty} \|U(t)\|_{H^s} = +\infty \,.$$

• The proof of the above result is a combination of a modified scattering on a product space with an analysis of a resonant system initiated in a work by Colliander-Keel-Staffilani-Takaoka-Tao. By using a refinement by Guardia-Kaloshin, we can give some quantification of the growth (for instance faster than any power of $\log\log(t)$).

The general problematic : NLS on product spaces

• Consider the Cauchy problem

$$(i\partial_t + \Delta - |U|^2)U = 0, \quad U|_{t=0} = U_0, \tag{2}$$

where now $U(t) : \mathbb{R}^n \times M \to \mathbb{C}$, M being a compact riemannian manifold.

• Let $n \geq 2$. Using a vector valued Strichartz estimate (i.e. exploiting only the x dispersion) we can obtain that for every U_0 which is small in a suitable Sobolev space, there is a unique global solution $U(t)$ (in a suitable class) of (2), there is a function V_0 in the initial data class such that

$$U(t) = e^{it\Delta}(V_0) + o(1), \quad t \gg 1,$$

in the initial data norm.

• For $n = 1$ one expects a modified scattering, i.e. the free evolution should be replaced by a dynamics with a more involved asymptotic behavior.

The case $n = 1$. Introduction of the resonant system

- From now on, we will only consider the case $n = 1$ and M a torus.
- Consider therefore the Cauchy problem

$$(i\partial_t + \Delta - |U|^2)U = 0, \quad U|_{t=0} = U_0, \tag{3}$$

where $U(t) : \mathbb{R} \times \mathbb{T}^d \to \mathbb{C}$, $d = 1, 2, 3, 4$.
Consider the "resonant part" of (3)

$$i\partial_t G(t) = \mathcal{R}[G(t), G(t), G(t)],$$

where the nonlinearity is given by

$$\mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} \mathcal{R}[G, G, G](\xi, p) = \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2=|q|^2+|s|^2}} \widehat{G}(\xi, q)\overline{\widehat{G}(\xi, r)}\widehat{G}(\xi, s).$$

$\widehat{G}(\xi, p) = \mathcal{F}_{\mathbb{R}\times\mathbb{T}^d} G(\xi, p)$ is the Fourier transform of G at $(\xi, p) \in \mathbb{R} \times \mathbb{Z}^d$.
- The dependence on ξ is merely parametric.

Norms

• We define a week norm

$$\|F\|_Z^2 := \sup_{\xi\in\mathbb{R}} \left[1+|\xi|^2\right]^2 \sum_{p\in\mathbb{Z}^d} [1+|p|]^2 \, |\widehat{F}(\xi,p)|^2 \, .$$

Z **is a conserved quantity for the resonant system.** It is expected that $e^{-it\Delta}U(t)$ remains bounded in Z.

• Fix $N \geq 30$. We define a strong norm

$$\|F\|_S := \|F\|_{H^N_{x,y}} + \|xF\|_{L^2_{x,y}}$$

and an even stronger one (but only in x !)

$$\|F\|_{S+} := \|F\|_S + \|(1-\partial_{xx})^4 F\|_S + \|xF\|_S.$$

• An important feature dictating the choice of the S and S^+ norms is the following property they should satisfy : if F is supported in $\{x : |x| > t^\alpha\}$, $\alpha > 0$ then there exists $\beta > 0$ such that

$$\|F\|_S \lesssim t^{-\beta} \|F\|_{S+}.$$

A similar property should hold if F contains only x frequencies $\gtrsim t^\alpha$.

Norms (sequel)

• The resonant system is globally well-posed for data in S and S^+ (the Z norm is conserved).

• For data in S^+, the solution of the full problem is expected to grow slightly in S^+. The difference between the true solution and the solution of the resonant system is supposed to decay in S (after factorizing the free evolution).

• We have the basic dispersive bound

$$\|e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}} F\|_{L^\infty_x H^1_y} \lesssim (1+|t|)^{-\frac{1}{2}}\|F\|_Z + (1+|t|)^{-\frac{5}{8}}\|F\|_S .$$

This bound follows by "summing-up with respect to the transverse variable" the classical $1d$ dispersive bound

$$\|e^{it\partial_x^2} f\|_{L^\infty_x} \lesssim (1+|t|)^{-\frac{1}{2}}\|\widehat{f}\|_{L^\infty} + (1+|t|)^{-\frac{3}{4}}\|xf\|_{L^2} .$$

Statement of the main results 1. Modified scattering.

Theorem 3 (HPTV'2013)
Let $1 \leq d \leq 4$. *There exists* $\varepsilon > 0$ *such that if* $U_0 \in S^+$ *satisfies*

$$\|U_0\|_{S^+} \leq \varepsilon,$$

and if $U(t)$ *solves the cubic defocusing NLS posed on* $\mathbb{R} \times \mathbb{T}^d$ *with initial data* U_0, *then* $U \in C((0, +\infty); S)$ *exists globally and exhibits modified scattering to its resonant dynamics in the following sense: there exists* $G_0 \in S$ *such that if* $G(t)$ *is the solution of the resonant system with initial data* $G(0) = G_0$, *then*

$$\|e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}} U(t) - G(\pi \ln t)\|_S \to 0 \qquad \text{as} \qquad t \to +\infty.$$

Moreover

$$\|U(t)\|_{L^\infty_x H^1_y} \lesssim (1 + |t|)^{-\frac{1}{2}}.$$

Statement of the main results 2. Existence of a wave operator.

Theorem 4 (HPTV'2013)

Let $1 \leq d \leq 4$. There exists $\varepsilon > 0$ such that if $G_0 \in S^+$ satisfies

$$\|G_0\|_{S^+} \leq \varepsilon,$$

and $G(t)$ solves the resonant system with initial data G_0, then there exists $U \in C((0,\infty); S)$ a solution of the cubic defocusing NLS, posed on $\mathbb{R} \times \mathbb{T}^d$ such that

$$\|e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}} U(t) - G(\pi \ln t)\|_S \to 0 \ \text{ as } \ t \to +\infty.$$

In particular

$$\|U(t) - e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}} G(\pi \ln t)\|_{H^N(\mathbb{R}\times\mathbb{T}^d)} \to 0 \ \text{ as } \ t \to +\infty.$$

Solutions with growing Sobolev norms

• We take initial data of the resonance system of the form

$$G_0(x,y) = \varepsilon \mathcal{F}_{\mathbb{R}}^{-1}(\varphi)(x)g(y), \quad x \in \mathbb{R}, y \in \mathbb{T}^d,$$

with φ real valued. The solution $G(t)$ to the resonance system with initial data $G_0(x,y)$ as above is given in Fourier space by

$$\widehat{G}_p(t,\xi) = \varphi(\xi) a_p(\varphi(\xi)^2 t), \quad a_p(0) = \mathcal{F}_{\mathbb{T}^d}(g)(p),$$

where the vector $a = (a_p)_{p \in \mathbb{Z}^d}$ solves the *resonant equation*

$$i\partial_t a_p(t) = \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2=|q|^2+|s|^2}} a_q(t)\overline{a_r(t)}a_s(t)\,.$$

• In particular, if $\varphi = 1$ on an open interval I, then $\widehat{G}_p(t,\xi) = a_p(t)$ for all $t \in \mathbb{R}$ and $\xi \in I$. We can therefore apply the following result which follows from some elaborations on the works by Colliander-Keel-Staffilani-Takaoka-Tao and Guardia-Kaloshin.

Solutions with growing Sobolev norms (sequel)

Theorem 5 (growth for the resonant equation)
Let $d \geq 2$ and $s > 1$. There exists global solutions to the resonant equation in $C(\mathbb{R}; h^s)$ such that

$$\sup_{t>0} \|a(t)\|_{h^s} = \infty.$$

More precisely, for any $\varepsilon > 0$, there exists a solution $a(t) \in C(\mathbb{R}; h^s)$ such that for some sequence of times $t_k \to \infty$ we have that

$$\|a(0)\|_{h^s} \leq \varepsilon, \qquad \|a(t_k)\|_{h^s} \gtrsim \exp(c(\log t_k)^{\frac{1}{2}})$$

for some $c > 0$.

- **Notation** :

$$\|(a_p)\|_{h^s}^2 := \sum_{p \in \mathbb{Z}^d} \left[1 + |p|^2\right]^s |a_p|^2.$$

- Unfortunately, $a(t) \notin h^\sigma$ for $\sigma > s$.

On the proof of the main results. Decomposition of the nonlinearity.

• Let $U(t)$ be a solution of the cubic defocusing NLS, posed on $\mathbb{R}\times\mathbb{T}^d$. Then $F(t) = e^{-it\Delta}U(t)$ solves

$$i\partial_t F(t) = \mathcal{N}^t[F(t), F(t), F(t)],$$

where the trilinear form $\mathcal{N}^t$ is defined by

$$\mathcal{N}^t[F,G,H] := e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}}\Big(e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}}F \cdot e^{-it\Delta_{\mathbb{R}\times\mathbb{T}^d}}\overline{G} \cdot e^{it\Delta_{\mathbb{R}\times\mathbb{T}^d}}H\Big).$$

• Now, we can compute the Fourier transform of the last expression which leads to the identity

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,p) = \sum_{p-q+r-s=0} e^{it\left[|p|^2-|q|^2+|r|^2-|s|^2\right]}\mathcal{I}^t[\widehat{F_q, G_r, H_s}](\xi),$$

where

$$\mathcal{I}^t[f,g,h] := \mathcal{U}(-t)\Big(\mathcal{U}(t)f\,\overline{\mathcal{U}(t)g}\,\mathcal{U}(t)h\Big), \quad \mathcal{U}(t) = \exp(it\partial_x^2).$$

Decomposition of the nonlinearity (sequel)

- One verifies that

$$\mathcal{I}^t\widehat{[f,g,h]}(\xi) = \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{f}(\xi-\eta)\overline{\widehat{g}}(\xi-\eta-\kappa)\widehat{h}(\xi-\kappa)d\kappa d\eta.$$

Thus one may also write

$$\mathcal{F}\mathcal{N}^t[F,G,H](\xi,p) = \sum_{p-q+r-s=0} e^{it\left[|p|^2-|q|^2+|r|^2-|s|^2\right]} \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{F_q}(\xi-\eta)\overline{\widehat{G_r}}(\xi-\eta-\kappa)\widehat{H_s}(\xi-\kappa)d\kappa d\eta\,.$$

A formal stationary phase argument ($t \gg 1$) suggests to define $\mathcal{R}$ as

$$\mathcal{F}\mathcal{R}[F,G,H](\xi,p) := \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2=|q|^2+|s|^2}} \widehat{F_q}(\xi)\overline{\widehat{G_r}}(\xi)\widehat{H_s}(\xi),$$

One expects that the nonlinearity can be decomposed as follows

$$\mathcal{N}^t[F,G,H] = \frac{\pi}{t}\mathcal{R}[F,G,H] + \text{ better term}$$

Recall that the resonant system is precisely $i\partial_t F = \mathcal{R}[F,F,F]$.

Decomposition of the nonlinearity (sequel)

• We have a remarkable Leibniz rule for $\mathcal{I}^t[f,g,h]$, namely

$$Z\mathcal{I}^t[f,g,h] = \mathcal{I}^t[Zf,g,h] + \mathcal{I}^t[f,Zg,h] + \mathcal{I}^t[f,g,Zh], \qquad Z \in \{ix, \partial_x\}.$$

A similar property holds for the whole nonlinearity $\mathcal{N}^t[F,G,H]$, where Z can also be ∂_{y_j}. This property is the analogue of the Klainerman vector fields relations in similar problems for the wave equation.

• The basic strategy in estimating the nonlinearity is to use $1d$ dispersive estimates for fixed frequencies of the periodic variable and then sum-up the pieces. In many cases (when we have S norms as outputs), we use the simple but useful bound

$$\Big\| \sum_{p-q+r-s=0} c_q^1 c_r^2 c_s^3 \Big\|_{l_p^2} \lesssim \min_{\sigma \in \mathfrak{S}_3} \|c^{\sigma(1)}\|_{l_p^2} \|c^{\sigma(2)}\|_{l_p^1} \|c^{\sigma(3)}\|_{l_p^1}.$$

In the remaining cases (with Z norm as an output), we use **multilinear Strichartz estimates on the torus**, in order to sum-up the pieces.

A basic bound

• Using the last inequality, the energy bound

$$\|\mathcal{I}^t[f^a, f^b, f^c]\|_{L^2_x} \lesssim \min_{\sigma\in\mathfrak{S}_3} \|f^{\sigma(a)}\|_{L^2_x}\|e^{it\partial_{xx}} f^{\sigma(b)}\|_{L^\infty_x}\|e^{it\partial_{xx}} f^{\sigma(c)}\|_{L^\infty_x}.$$

and the dispersive bound

$$\|e^{it\partial_{xx}} f\|_{L^\infty_x} \lesssim |t|^{-\frac{1}{2}}\|f\|^{\frac{1}{2}}_{L^2_x}\|xf\|^{\frac{1}{2}}_{L^2_x},$$

we get the following basic bound

$$\|\mathcal{N}^t[F, G, H]\|_S \lesssim (1 + |t|)^{-1}\|F\|_S\|G\|_S\|H\|_S.$$

• Therefore, being optimistic we may hope to apply modified scattering techniques. The last bound is not very useful alone but it may become sufficient if one of the functions F, G, H has a better decay, for instance if it is localized at high frequencies (in terms of $t \gg 1$) or away of the origin in the physical space (again in terms of t, e.g the region $|x| > t^{\frac{1}{100}}$).

A key proposition

$$\|F\|_{X_T} := \sup_{0\le t\le T}\Big(\|F(t)\|_Z + \langle t\rangle^{-\delta}\|F(t)\|_S + \langle t\rangle^{1-3\delta}\|\partial_t F(t)\|_S\Big),$$
$$\|F\|_{X_T^+} := \|F\|_{X_T} + \sup_{0\le t\le T}\Big(\langle t\rangle^{-5\delta}\|F(t)\|_{S+} + \langle t\rangle^{1-7\delta}\|\partial_t F(t)\|_{S+}\Big),$$

where $\delta \in (0, 10^{-3})$ is fixed. We have the following key statements.

Proposition 6

For $T \ge 1$, we can decompose the nonlinearity as

$$\mathcal{N}^t[F(t), G(t), H(t)] = (\frac{\pi}{t}\mathcal{R} + \mathcal{E}^t)[F(t), G(t), H(t)],$$

with the bounds

$$\|\int_{T/2}^{T}\mathcal{E}^t[F(t), G(t), H(t)]dt\|_S \lesssim T^{\delta}\|F\|_{X_T}\|G\|_{X_T}\|H\|_{X_T}$$

and

$$\|\int_{T/2}^{T}\mathcal{E}^t[F(t), G(t), H(t)]dt\|_Z \lesssim T^{-\delta}\|F\|_{X_T}\|G\|_{X_T}\|H\|_{X_T}$$

uniformly in $T \ge 1$.

A key proposition (sequel)

Proposition 7

In the context of the previous proposition, if we assume in addition

$$\|F\|_{X_T^+} + \|G\|_{X_T^+} + \|H\|_{X_T^+} \leq 1,$$

then we also have

$$\| \int_{T/2}^{T} \mathcal{E}^t[F(t), G(t), H(t)]dt\|_S \lesssim T^{-2\delta} .$$

• The second proposition has the spirit of the first one, where the couple (Z, S) is "lifted" to (S, S^+).

On the proof of the key proposition I. A first reduction.

• We perform the decomposition of the nonlinearity

$$\sum_{A,B,C-\text{dyadic}} \mathcal{N}^t[Q_A F(t), Q_B G(t), Q_C H(t)],$$

where Q_A, Q_B, Q_C are Littlewood-Paley projectors in the x variable.

• If we look for decay estimates for $t \sim T$, $T \gg 1$, then in the regime $\max(A, B, C) \geq T^{\frac{1}{6}}$ we can exchange frequency localization to decay in time thanks to the bilinear refinements of the Strichartz estimate on $\mathbb{R}$. The summation in the y frequencies is done via the rough bound.

• Thus we may suppose that the x frequencies of F, G, H are $\lesssim T^{\frac{1}{6}}$.

On the proof of the key proposition II. The fast time oscillations.

• In order to make a second reduction, we split the nonlinearity as

$$\mathcal{N}^t[F,G,H] = \Pi^t[F,G,H] + \widetilde{\mathcal{N}}^t[F,G,H],$$

with

$$\mathcal{F}\widetilde{\mathcal{N}}^t[F,G,H](\xi,p) = \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2\neq|q|^2+|s|^2}} e^{it\left[|p|^2-|q|^2+|r|^2-|s|^2\right]}\mathcal{I}^t[\widehat{F_q, G_r}, H_s](\xi).$$

• Recall that

$$\mathcal{I}^t[\widehat{f,g,}h](\xi) = \int_{\mathbb{R}^2} e^{it2\eta\kappa}\widehat{f}(\xi-\eta)\overline{\widehat{g}}(\xi-\eta-\kappa)\widehat{h}(\xi-\kappa)d\kappa d\eta.$$

• To bound $\widetilde{\mathcal{N}}^t[F,G,H]$, we distinguish two cases :

1. If $|\eta\kappa| \lesssim T^{-\frac{1}{4}}$ then we integrate by parts in t thanks to the oscillation $e^{it(|p|^2-|q|^2+|r|^2-|s|^2)}$ (normal form reduction).

2. If $|\eta\kappa| \gtrsim T^{-\frac{1}{4}}$ then we integrate by parts in κ, thanks to the oscillation $e^{it2\eta\kappa}$, since on the support of the integration $|\eta| \gtrsim T^{-\frac{5}{12}}$.

On the proof of the key proposition III. The resolvent level set.

- In this key place of the analysis, we use the following bound :

$$\|R[a^1,a^2,a^3]\|_{l^2_p} \le C_d \min_{\tau\in\mathfrak{S}_3} \|a^{\tau(1)}\|_{l^2_p}\|a^{\tau(2)}\|_{h^1_p}\|a^{\tau(3)}\|_{h^1_p}$$

(R is $\mathcal{R}$ liberated from the ξ dependence). The proof of this bound uses multi-linear Strichartz estimates on the torus (particularly hard for $d = 4$).

- Set

$$\|F\|_{\tilde{Z}_t} := \|F\|_Z + (1+|t|)^{-\delta}\|F\|_S,$$

Proposition 8 (the main estimate)

We have

$$\|\Pi^t[F^a,F^b,F^c]\|_S \lesssim (1+|t|)^{-1} \sum_{\sigma\in\mathfrak{S}_3} \|F^{\sigma(a)}\|_{\tilde{Z}_t}\cdot\|F^{\sigma(b)}\|_{\tilde{Z}_t}\cdot\|F^{\sigma(c)}\|_S$$

and

$$\|\Pi^t[F,G,H] - \frac{\pi}{t}\mathcal{R}[F,G,H]\|_S \lesssim (1+|t|)^{-1-20\delta}\|F\|_{S+}\|G\|_{S+}\|H\|_{S+}.$$

The resolvent level set (sequel).

• Let us give the proof of the first part of the above proposition : By a soft argument, we estimate :

$$\|\Pi^t[F^a, F^b, F^c](x)\|_{L^2_{x,y}}$$

by

$$C\Big\| \sum_{\substack{p+r=q+s \\ |p|^2+|r|^2=|q|^2+|s|^2}} |e^{it\partial_{xx}} F^a_q(x)| \cdot |e^{-it\partial_{xx}} F^b_r(x)| \cdot |e^{it\partial_{xx}} F^c_s(x)| \Big\|_{L^2_{x,p}}$$

and by the Strichartz bound, we can continue as follows

$$\lesssim \min_{j\in\{a,b,c\}} \|e^{it\partial_{xx}} F^j_p(x)\|_{L^2_{x,p}} \prod_{k\neq j} \left[\sup_x \sum_{p\in\mathbb{Z}^d} \left[1+|p|^2\right] |e^{it\partial_{xx}} F^k_p(x)|^2 \right]^{\frac{1}{2}}$$

Applying **an abstract transverse principle**, we deduce the claimed estimated in S thanks to our dispersive bound

$$\sup_{x\in\mathbb{R}} \sum_{p\in\mathbb{Z}^d} \left[1+|p|^2\right] |e^{it\partial_{xx}} F_p(x)|^2 \lesssim \langle t\rangle^{-1}\Big(\|F\|^2_Z + \langle t\rangle^{-\frac{1}{4}}\|F\|^2_S\Big).$$

The resolvent level set (sequel of the sequel).

- For the second estimate, we use in addition a soft stationary phase argument.
- The Strichartz argument also gives the bound

$$\|\mathcal{R}[F^a, F^b, F^c]\|_Z \lesssim \|F^a\|_Z \|F^b\|_Z \|F^c\|_Z.$$

Construction of the modified wave operator

• The existence of a modified wave operator now follows by a fix point argument for

$$F(t) \mapsto -i \int_t^\infty \left\{ \mathcal{N}^\sigma[F+G, F+G, F+G] - \frac{\pi}{\sigma}\mathcal{R}[G(\sigma), G(\sigma), G(\sigma)] \right\} d\sigma$$

Thanks to our estimates in S^+, we have that

$$\int_t^\infty \mathcal{E}^\sigma[G(\sigma), G(\sigma), G(\sigma)] d\sigma, \qquad G(t) \in S^+,$$

decays like $(1+|t|)^{-\delta}$ in S and like $(1+t)^{-2\delta}$ in Z.

• Thanks to our estimates, we can reproduce this information and construct F. Namely

$$(1+t)\|\mathcal{N}^t[F(t), G(t), G(t)]\|_Z \lesssim \|G(t)\|_Z^2 \|F(t)\|_Z + \text{better}$$

and

$$(1+t)\|\mathcal{N}^t[F(t), G(t), G(t)]\|_S \lesssim \|G\|_{\tilde{Z}_t}^2 \|F\|_S + \|F\|_{\tilde{Z}_t}\|G\|_{\tilde{Z}_t}\|G\|_S + \text{better}$$

(recall that $\|F\|_{\tilde{Z}_t} := \|F\|_Z + (1+|t|)^{-\delta}\|F\|_S$).
One can conclude.

Modified scattering

- The proof of the modified scattering statement follows similar lines.
- Roughly speaking one gets bounds in the strong norm S^+ and convergence in the weaker norm S.
- There is however an important additional ingredient concerning the estimates of the solutions of

$$\partial_t F(t) = \mathcal{N}^t[F(t), F(t), F(t)] = (\frac{\pi}{t}\mathcal{R} + \mathcal{E}^t)[F(t), F(t), F(t)] \qquad (4)$$

in the norm Z. For that purpose one multiplies (4) with the multiplier giving the Z conservation of the resonant system. This allows to get rid of the singular term in the right hand-side of (4).

- Consequently, even if we have a small data result its proof is not perturbative since it uses the conservation law of the resonant system.

Final comments

• A very nice reference concerning the modified scattering for the cubic NLS on $\mathbb{R}$ is the work by Kato-Pusateri, where one finds a new proof of the classical result by Ozawa.

• One may wish to see the result as a sort of transverse stability of the resonant dynamics.

• One may obtain similar modified scattering results if $\mathbb{T}^d$, $d = 1, 2, 3, 4$ is replaced by S^d, $d = 2, 3$ or $\mathbb{T} \times S^2$. Therefore the understanding of the corresponding resolvent equations is an interesting issue. A similar comment applies for NLS with a partial harmonic confinement.

• Cover the whole range $s > 1$ is a remaining issue. It looks that one should further elaborate on the growing norm mechanism for the resonant system ...