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Solving nonlinear PDE's in the presence of low regularity randomness

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Introduction

• Our goal is to describe results concerning the construction of *low regularity* solutions of partial differential equations, depending on a random parameter.

• The motivations for these studies are multiple. However, at the end, the obtained results and the methods leading to these results are conceptually close to each other.

Multiple Fourier series

- Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be the torus of dimension d.
- If $f : \mathbb{T}^d \to \mathbb{C}$ is a C^∞ function then for every $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n)\, e^{in\cdot x},$$

where $\widehat{f}(n)$ are the Fourier coefficients of f, defined by

$$\widehat{f}(n) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-in\cdot x} dx\,.$$

Sobolev spaces on the torus

• For $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we set

$$\langle x \rangle := (1 + x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}.$$

• For $s \in \mathbb{R}$, we define the Sobolev norm of f by

$$\|f\|_{H^s(\mathbb{T}^d)}^2 = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\widehat{f}(n)|^2. \tag{1}$$

• For $s \geq 0$ an integer, we have the norm equivalence

$$\|f\|_{H^s(\mathbb{T}^d)}^2 \approx \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2(\mathbb{T}^d)}^2. \tag{2}$$

In (2), ∂^α denotes a partial derivative of order at most s.

• For $s = 0$, we recover the Lebesgue space $L^2(\mathbb{T}^d)$.

• The Sobolev space $H^s(\mathbb{T}^d)$ is defined as the closure of $C^\infty(\mathbb{T}^d)$ with respect to the norm (1).

$\boxed{\text{Probabilistic effects in problems of fine analysis}}$

- We first discuss *an almost sure improvement* of the Sobolev embedding.
- We say that a random variable g belongs to $\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$, if $g = h + il$, where $h \in \mathcal{N}(0,\sigma^2)$ and $l \in \mathcal{N}(0,\sigma^2)$ are independent.
- Let $u \in L^2(\mathbb{T})$ be a deterministic function. There is a sequence $(c_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ (the Fourier coefficients of u) such that

$$u(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}\,.$$

- Consider now a randomised version of u given by the expression

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} c_n\, g_n(\omega)\, e^{inx}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ are independent from $\mathcal{N}_{\mathbb{C}}(0,1)$.

$\boxed{\text{Almost sure improvement of the Sobolev embedding}}$

• We have $g_n(\omega)\, e^{inx} \in \mathcal{N}_{\mathbb{C}}(0,1)$ (invariance under rotations of $\mathcal{N}_{\mathbb{C}}(0,1)$).

• Next, using the independence of g_n we get that for a fixed $x \in \mathbb{T}$,

$$u_\omega(x) \in \mathcal{N}_{\mathbb{C}}\Big(0,\ \sum_{n\in\mathbb{Z}} |c_n|^2\Big)\,.$$

• Consequently $u_\omega(x) \in L^p(\Omega \times \mathbb{T}),\ \forall\, p < \infty$ which gives :
For every $p < \infty$, $u_\omega(x) \in L^p(\mathbb{T})$, almost surely.

• The last statement is to compare with the Sobolev embedding :
$H^{\frac{1}{2}}(\mathbb{T})$ is continuously embedded in $L^p(\mathbb{T})$ for every $p < \infty$. The statement is false, if we replace $H^{\frac{1}{2}}(\mathbb{T})$ with $H^s(\mathbb{T})$ for some $s < 1/2$.

$\boxed{\text{Remarks}}$

• Informally : the randomisation gains a $1/2$ derivative.

• We can replace the gaussians with much more general random variables (Bernoulli variables, for instance).

Products in Sobolev spaces of negative indexes

• Consider the random series :

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha}\, e^{inx}, \qquad \frac{1}{4} < \alpha < \frac{1}{2}\,,$$

with g_n as in the previous discussion.

• We have that a.s. $u_\omega \in H^\sigma(\mathbb{T})$, $\sigma < \alpha - \frac{1}{2}$ but a.s. $u_\omega \notin H^{\alpha-\frac{1}{2}}(\mathbb{T})$.

• Fix $\sigma < \alpha - \frac{1}{2}$ (close to $\alpha - \frac{1}{2}$).

• u_ω belongs to a Sobolev space of negative regularity and therefore it is hard to define an object like $|u_\omega|^2$.

• It is however possible, *after a renormalisation*, to define $|u_\omega|^2$ and even compute its Sobolev regularity.

Products in Sobolev spaces of negative indexes

• Consider the partial sums

$$u_{\omega,N}(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx} \in C^\infty(\mathbb{T})$$

and write

$$|u_{\omega,N}(x)|^2 = \sum_{|n|\leq N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2 \\ |n_1|,|n_2|\leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha \langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}.$$

• The first term (the zero Fourier coefficient) contains all the singularity while the second has an a.s. limit in $H^{2\sigma}(\mathbb{T})$.

Products in Sobolev spaces of negative indexes

- Consequently, we set

$$c_N := \mathbb{E}\Big(\sum_{|n| \leq N} \frac{|g_n(\omega)|^2}{\langle n \rangle^{2\alpha}} \Big) = \sum_{|n| \leq N} \frac{2}{\langle n \rangle^{2\alpha}} \sim N^{1-2\alpha},$$

and we define the renormalised partial sums

$$|u_{\omega,N}(x)|^2 - c_N = \sum_{|n| \leq N} \frac{|g_n(\omega)|^2 - 2}{\langle n \rangle^{2\alpha}} + \sum_{\substack{n_1 \neq n_2 \\ |n_1|,|n_2| \leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1 \rangle^{\alpha} \langle n_2 \rangle^{\alpha}} \, e^{i(n_1 - n_2)x}.$$

- Thanks to the independence of g_n we have

$$\mathbb{E}\Big(\Big| \sum_{|n| \leq N} \frac{|g_n(\omega)|^2 - 2}{\langle n \rangle^{2\alpha}} \Big|^2 \Big) = \sum_{|n| \leq N} \frac{4}{\langle n \rangle^{4\alpha}},$$

which has a limit as $N \to \infty$ when $\alpha > 1/4$.

Products in Sobolev spaces of negative indexes

- Another use of the independence yields that

$$\mathbb{E}\Big(\Big\| \sum_{\substack{n_1 \neq n_2 \\ |n_1|,|n_2| \leq N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^{\alpha}\langle n_2\rangle^{\alpha}} \, e^{i(n_1-n_2)x} \Big\|^2_{H^{2\sigma}}\Big)$$

is bounded by

$$C \sum_{n_1,n_2} \frac{\langle n_1 - n_2\rangle^{4\sigma}}{\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}} \, .$$

The last sum is convergente as far as $-4\sigma + 4\alpha > 2$, which is equivalent to our assumption $\sigma < \alpha - \frac{1}{2}$.

- Hence the sequence

$$\Big(|u_{\omega,N}(x)|^2 - c_N\Big)_{N \geq 1}$$

has a limit in $L^2(\Omega; H^{2\sigma}(\mathbb{T}))$. This limit is by definition the renormalisation of $|u_\omega|^2$.

Remarks

• Using more involved arguments, we can also show the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T})$ of the sequence

$$\Big(|u_{\omega,N}(x)|^2 - c_N\Big)_{N\geq 1}\,.$$

• Since $\sigma < 0$ the norm in $H^{2\sigma}(\mathbb{T})$ is weaker than in $H^{\sigma}(\mathbb{T})$ (where $u_\omega(x)$ is defined).

• Informally : the square of the modulus of an element of H^σ is in $H^{2\sigma}$, after a renormalisation.

• This is a remarkable probabilistic phenomenon, in the heart of the study of evolution partial differential equations in the presence of randomness in Sobolev spaces of negative indexes.

The nonlinear wave equation

Theorem 1 (classical)

• *For every* $(u_0, u_1) \in H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ *there exists a unique global solution of*

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x)$$

in the class $(u, \partial_t u) \in C(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$.

• *If in addition* $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ *for some* $s \geq 1$ *then*

$$(u, \partial_t u) \in C(\mathbb{R}; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)) .$$

The dependence with respect to the initial data is continuous.

• The local in time part of Theorem 1 can be extended to the case $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$, $s \geq 1/2$, and the global in time part to $s > 3/4$ (Kenig-Ponce-Vega, Gallagher-Planchon, Roy).

• We conjecture that Theorem 1 remains true for $s \geq 1/2$ (proved recently by Dodson in the radial case of $\mathbb{R}^3$).

Limit of the deterministic methods

Theorem 2
Let $s \in (0, 1/2)$ *et* $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. *There exists a sequence*

$$u_N(t, x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \quad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta)u_N + u_N^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N(0) - u_0, \partial_t u_N(0) - u_1)\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \sup_{0 \le t \le T} \|u_N(t)\|_{H^s(\mathbb{T}^3)} = +\infty.$$

• The proof is based on an idea introduced by Gilles Lebeau and further developed by Christ-Colliander-Tao, Burq-Tz., Xia.

Solving the equation by probabilistic methods

• We can ask whether some form of well-posedness survives for initial data in

$$H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3), \quad s < 1/2. \tag{3}$$

• The answer of this question is positive if we endow the space (3) with a non degenerate probability measure such that we have the existence, the uniqueness, and a form of continuous dependence almost surely with respect to this measure.

Choice of the measure

• We will choose the initial data among the realisations of the following random series

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\,, \qquad u_1^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,. \tag{4}$$

Here $\{g_n\}_{n\in\mathbb{Z}^3}$ et $\{h_n\}_{n\in\mathbb{Z}^3}$ are two families of independent random variables conditioned by $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$, so that u_0^ω and u_1^ω are real valued.

• In addition, we suppose that for $n \neq 0$, g_n and h_n are complex gaussians from $\mathcal{N}_{\mathbb{C}}(0,1)$, and that g_0 and h_0 are standard real gaussians from $\mathcal{N}(0,1)$.

• The initial data (4) belong almost surely to $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. Moreover, the probability of the event

$$(u_0^\omega, u_1^\omega) \in H^{\alpha-\frac{3}{2}}(\mathbb{T}^3) \times H^{\alpha-\frac{5}{2}}(\mathbb{T}^3)$$

is zero.

Reformulation of the ill-posedness result

Theorem 3

Let $\alpha \in (3/2, 2)$ *and* $0 < s < \alpha - 3/2$. *For almost every* ω, *there exists a sequence*

$$u_N^\omega(t,x) \in C^\infty(\mathbb{R} \times \mathbb{T}^3), \qquad N = 1, 2, \cdots$$

such that

$$(\partial_t^2 - \Delta)u_N^\omega + (u_N^\omega)^3 = 0$$

with

$$\lim_{N \to +\infty} \|(u_N^\omega(0) - u_0^\omega, \partial_t u_N^\omega(0) - u_1^\omega)\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for every $T > 0$,

$$\lim_{N \to +\infty} \sup_{0 \le t \le T} \|u_N^\omega(t)\|_{H^s(\mathbb{T}^3)} = +\infty.$$

We can however prove the following result:

Theorem 4 (Burq-Tz. (2010))
Let $\alpha \in (3/2, 2)$ and $0 < s < \alpha - 3/2$. Define (thanks to the classical well-posedness result) the sequence $(u_N)_{N\geq 1}$ of solutions of

$$(\partial_t^2 - \Delta)u + u^3 = 0 \tag{5}$$

with C^∞ initial data

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}\,, \qquad u_1^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}\,.$$

The sequence $(u_N)_{N\geq 1}$ converges almost surely as $N \to \infty$ in $C(\mathbb{R}; H^s(\mathbb{T}^3))$ to a (unique) limit u which satisfies (5) in the distributional sense.

- The type of the approximation of the initial data is crucial when we prove probabilistic low regularity well-posedness.
- Even if we consider the approximation of the initial data by Fourier truncation there is dense set of pathological data such that the statement of Theorem 4 does not hold (very recent work by Sun-Tz.).
- We can prove uniqueness in a suitable functional framework.
- We can consider more general randomisations (this fact had an important impact in the field).

Going further

Theorem 5 (Oh-Pocovnicu-Tz. (2019))
Let $\alpha \in (\frac{5}{4}, \frac{3}{2}]$ and $s < \alpha - 3/2$. There exist positive constants γ, c, C, T_0 and a divergent sequence $(c_N)_{N\geq 1}$ such that for every $T \in (0, T_0)$ there exists a set Ω_T of complemental probability $\leq C \exp(-c/T^\gamma)$ such that if we denote by $(u_N^\omega)_{N\geq 1}$ the solution of

$$\partial_t^2 u - \Delta u - c_N u + u^3 = 0, \tag{6}$$

with initial data given by

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \qquad u_{1,N}^\omega(x) = \sum_{|n|\leq N} \frac{h_n(\omega)}{\langle n\rangle^{\alpha-1}} e^{in\cdot x}$$

then for every $\omega \in \Omega_T$ the sequence $(u_N^\omega)_{N\geq 1}$ converges as $N \to \infty$ in $C([-T, T]; H^s(\mathbb{T}^3))$. In particular, for almost every ω there exists $T_\omega > 0$ such that $(u_N^\omega)_{N\geq 1}$ converges in $C([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$.

- Theorem 5 in the first step in the study of the nonlinear wave equation in Sobolev spaces of negative indexes.
- The ultimate goal is to arrive to $\alpha = 1$ ...

Invariant measures for the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u - |u|^2 u = 0, \quad u(0,x) = u_0(x) \quad x \in \mathbb{T}^2. \tag{7}$$

• (7) is a Hamiltonian PDE. Therefore

$$E(u) = \int_{\mathbb{T}^2} \Big(|\nabla_x u(t,x)|^2 + |u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4 \Big) dx$$

is a (formally) conserved quantity for (7).

• The Gibbs measure associated with (7) is a **renormalisation** of the completely formal object

$$\exp(-E(u))du\,.$$

• The measure obtained by this **renormalisation** is absolutely continuous with respect to the gaussian measure given by the random series

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a family of independent (complex valued) gaussians from $\mathcal{N}_{\mathbb{C}}(0,1)$.

Theorem 6 (Bourgain (1996))

• *Let* $(u_N^\omega)_{N\geq 1}$ *be the sequence of solutions of*

$$(i\partial_t + \Delta)u - |u|^2 u = 0 \tag{8}$$

with C^∞ *initial data given by*

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}\,.$$

For every $s < 0$*, the sequence*

$$\left(\exp\left(\frac{it}{2\pi^2}\,\|u_N^\omega(t)\|_{L^2}^2\right) u_N^\omega(t)\right)_{N\geq 1}$$

converges almost surely in $C(\mathbb{R}; H^s(\mathbb{T}^2))$ *to a limit which satisfies a renormalised version of* (8)*.*

• *Moreover, the Gibbs measure is invariant under the resulting flow.*

Remarks

• The statement of the results by Bourgain and Burq-Tz. are similar. A notable difference is that in the Bourgain theorem, in order to obtain a limit one needs to reanormalise the sequence of approximate solutions $(u_N^\omega)_{N\geq 1}$. Moreover in Bourgain's theorem the randomisation is "rigid".

• We can formulate the Bourgain theorem in the spirit of the result by Oh-Pocovnicu-Tz. More precisely, one can prove the convergence of the solutions of

$$i\partial_t u + \Delta u + c_N u - |u|^2 u = 0$$

with data

$$u_0^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n \rangle} e^{in\cdot x},$$

where $(c_N(\omega))_{N\geq 1}$ is a sequence of real numbers almost surely divergent to $+\infty$.

Singular stochastic PDE's

• The problematic considered in the previous slides is close to the analysis of parabolic PDE's in the presence of a singular random source term (noise).
• The closest to the previously considered models is the nonlinear heat equation

$$\partial_t u - \Delta u + u^3 = \xi, \quad u(0,x) = 0 \quad x \in \mathbb{T}^3. \tag{9}$$

• Here ξ is the space-time white noise on $[0,\infty[\times\mathbb{T}^3$. It is the source term ξ which represents the singular randomness in (9) (in the previous slides it was the low regularity random initial data which represented the singular randomness).
• The white noise on $[0,\infty[\times\mathbb{T}^3$ may be written as

$$\xi = \sum_{n\in\mathbb{Z}^3} \dot{\beta}_n(t) e^{in\cdot x}, \tag{10}$$

where β_n are independent Brownian motions, conditioned by $\beta_n = \overline{\beta_{-n}}$ (β_0 is real and for $n \neq 0$, β_n is with values in $\mathbb{C}$).

Singular stochastic PDE's (sequel)

• For $N \gg 1$, an approximation of ξ by smooth functions is given by $\xi_N(t,x) = \rho_N \star \xi$ where $\rho_N(t,x) = N^5 \rho(N^2 t, Nx)$ with ρ a test function with integral 1 on $[0,\infty[\times \mathbb{T}^3$.

Theorem 7 (Hairer (2014), Mourrat-Weber (2018))
There is a sequence $(c_N)_{N\geq 1}$ *of positive numbers, divergent as* $N \to \infty$ *such that if we denote by* u_N *the solution of*

$$\partial_t u_N - \Delta u_N - c_N u_N + u_N^3 = \xi_N, \quad u(0,x) = 0$$

then $(u_N)_{N\geq 1}$ *converges in probability as* $N \to \infty$*.*

• We can also have almost sure convergence in suitable Hölder spaces. The initial data $u(0,x)$ can be different from zero : it suffices that it belongs to a suitable functional framework.

$\boxed{\text{Remarks}}$

• The result remains true for a noise ξ defined by

$$\xi = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}^3} g_{m,n}(\omega) e^{imt}\, e^{in\cdot x},$$

where $\{g_{m,n}\}_{(m,n)\in\mathbb{Z}^4}$ is a family of independent complex gaussians conditioned so that ξ is real values (white noise on $\mathbb{T}\times\mathbb{T}^3$).

• The two dimensional case is treated in the work by Da Prato-Debussche (2003).

• There are other parabolic PDE's for which one can obtain results in similar spirit, the most popular being probably the KPZ equation.

On the structure of the proofs

• The proofs of the previously described results follow the same scheme.

• First, we construct local in time solutions. Then we use a global information which is either an invariant measure or an energy estimate in order to get global in time solutions.

• In order to construct the solutions locally in time, we look for the solution in the form

$$u = u_1 + u_2,$$

where u_1 contains the singular part of the solution.

• Using probabilistic arguments, close the the ones in the beginning of the lecture, we prove that u_1 has properties better than the properties given by deterministic methods. All probabilistic part of the argument is in this part of the analysis.

• In the proof of the result by Burq-Tz. we use a.s. improvements of the Sobolev embedding while all the other results use products in Sobolev spaces of negative indexes.

On the structure of the proofs (sequel)

• We then solve the problem for u_2 by purely deterministic arguments. Here the nature of the equation becomes even more important. In the case of the heat equation, the basic tool is the elliptic regularity while for the other equations we exploit the time oscillations in a crucial way (these oscillations are captured by the Bourgain spaces, for instance).

• The passage from local to global solutions in the result by Bourgain uses an invariant measure as a global control on the solutions. In the result by Burq-Tz. the globalisation is done by energy estimates. It is remarkable that in the context of the nonlinear heat equation these two techniques are also used to globalise the solutions.

$\boxed{\text{A final remark}}$

- In the work by Burq-Tz. we allow more general randomisations compared to Bourgain's work. However, the proof does not say anything about the nature of the transported by the flow initial measure while in the work by Bourgain the initial gaussian measure is quasi-invariant under the flow.

- This fact motivated recent work on quasi-invariant measures for nonlinear dispersive equations.