Kyoto, July 19, 2013

Multi-solitons for the water-waves system

based on joint work with

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The equation

• We study the (infinite dimensional) Hamiltonian equation

$$\partial_t \eta = \frac{\delta E}{\delta \phi}, \qquad \partial_t \varphi = -\frac{\delta E}{\delta \eta}, \tag{1}$$

where

$$E(\eta,\varphi) = \frac{1}{2}\int_{-\infty}^{\infty}\Big[(G[\eta]\varphi)\,\varphi + g\eta^2 + 2b(\sqrt{1+(\partial_x\eta)^2} - 1)\Big]dx.$$

Here g is the gravity constant and b the surface tension coefficient.
• $G[\eta]$ denotes a Dirichlet-Neumann map defined as follows. For given "nice" $\eta,\varphi:\mathbb{R}\to\mathbb{R}$, s.t. $\inf_x \eta(x) > -H$, we define $\Phi(x,z)$ as

$$(\partial_x^2 + \partial_z^2)\Phi = 0, \quad \text{in} \quad -H < z < \eta(x),$$

with boundary conditions $\partial_z\Phi(x,-H) = 0, \Phi(x,\eta(x)) = \varphi(x)$. Then by definition

$$G[\eta]\varphi = \partial_z\Phi(x,\eta(x)) - \partial_x\eta(x)\partial_x\Phi(x,\eta(x)).$$

• The energy E is formally conserved by the flow of (1). So is the momentum : $\int \eta\,\partial_x\varphi$. These facts are of crucial importance for our analysis.

The solitary wave solutions

Theorem 1 (Amick-Kirchgässner)
Suppose that the speed c is such that

$$\frac{gH}{c^2} = 1 + \varepsilon^2, \qquad \frac{b}{Hc^2} > \frac{1}{3}.$$

Then there exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ (which fixes the speed c) there is a solution of the water-wave system under the form

$$Q_c(x - ct) = (\eta_c(x - ct), \varphi_c(x - ct))$$

where η_c and φ_c satisfy:

$$\exists\, d > 0, \quad \forall\, \alpha \geq 0, \quad \exists\, C_\alpha > 0, \quad \forall\, x \in \mathbb{R}, \quad |\partial_x^\alpha \eta_c(x)| \leq C_\alpha e^{-d|x|}$$

and

$$\exists\, d > 0, \quad \forall\, \alpha \geq 1, \quad \exists\, C_\alpha > 0, \quad \forall\, x \in \mathbb{R}, \quad |\partial_x^\alpha \varphi_c(x)| \leq C_\alpha e^{-d|x|}.$$

Moreover η_c is even and φ_c is odd.

Relation with the KdV solitary wave

In fact, we have that

$$Q_c(x) = (\eta_c(x), \varphi_c(x)) = (H\eta_\varepsilon(H^{-1}x),\, cH\varphi_\varepsilon(H^{-1}x))$$

with

$$\eta_\varepsilon(x) = \varepsilon^2\Theta_1(\varepsilon x, \varepsilon), \quad \varphi_\varepsilon(x) = \varepsilon\Theta_2(\varepsilon x, \varepsilon),$$

where Θ_1 and Θ_2 satisfy:

$$\exists\,\delta > 0,\, \forall\,\alpha \geq 0,\, \exists\,C_\alpha > 0,\, \forall\,(x,\varepsilon) \in \mathbb{R}\times(0,\varepsilon_0), \quad |(\partial_x^\alpha\Theta_1)(x,\varepsilon)| \leq C_\alpha e^{-\delta|x|}$$

and

$$\exists\,\delta > 0,\, \forall\,\alpha \geq 1,\, \exists\,C_\alpha > 0,\, \forall\,(x,\varepsilon) \in \mathbb{R}\times(0,\varepsilon_0), \quad |(\partial_x^\alpha\Theta_2)(x,\varepsilon)| \leq C_\alpha e^{-\delta|x|}.$$

The functions $\Theta_1(x,\varepsilon)$ and $\Theta_2(x,\varepsilon)$ have smooth expansions in ε. One has that

$$\Theta_1(\xi, 0) = -\cosh^{-2}\Big(\frac{\xi}{2(\beta - 1/3)^{1/2}}\Big),$$

which reminds us the KdV soliton.

Set

$$M(t,x) := Q_{c_1}(x - c_1 t) + Q_{c_2}(x - h - c_2 t)\,.$$

where $h > 0$.

Theorem 2 (http://arxiv.org/abs/1304.5263)

Let us fix $s \geq 0$. Suppose that the speeds $c_1 < c_2$ satisfy

$$\frac{gH}{c_j^2} = 1 + \varepsilon_j^2, \quad \frac{b}{Hc_j^2} > \frac{1}{3}, \quad j = 1,2.$$

Then there exists ε^* such that for $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon^*]$ and h sufficiently large, we have that there exists a (semi) global solution $U(t) = (\eta(t), \varphi(t))$, $t \geq 0$ to the water-wave system satisfying

$$U \in \mathcal{C}_b([0,\infty);\, H^s(\mathbb{R}) \times H^s(\mathbb{R}))$$

and

$$\lim_{t \to +\infty} \|U(t) - M(t)\|_{H^s \times H^s} = 0.$$

Comments

• This result is inspired by previous works on simplified model equations such as the KdV, NLS or Hartree equations. Most notably, the works by Merle (1990), Martel (2005), Krieger-Martel-Raphaël (2009) and Combet (2010).

• One can take $h = 0$ but then the solution is defined only on $[T_0, +\infty)$, $T_0 \gg 1$.

• The behavior of $U(t)$ for $t \leq 0$ is unclear. One may wish to expect that it is globally defined. In a regime $c_1 \sim c_2$ one may expect to perform the "finite box" analysis of the recent work by Martel-Merle.

• The uniqueness of $U(t)$ is unclear.

• Similarly, one can construct a k-soliton for $k > 2$.

Scheme of the proof

•The proof relies on the following ingredients :

1. Consistency of M with the equation.

2. Estimate for the possible growth of the linearization about M (a "stability" property).

3. Quasi-linear energy estimates.

• Ingredient 1 uses in an essential way the nature of the fluid domain.

• Ingredient 2 relies on a stability analysis by Mielke.

• Ingredient 3 was already done in a previous work with F. Rousset on the transverse stability of solitary waves.

• With the three ingredients in hand, one concludes by invoking the Grenier argument.

• This scheme seems to apply to all previously studied situations. We hope that it may be extended to the case without surface tension ($b = 0$). However in this case Ingredient 2 would rely on different arguments developed recently by Pego-Sun.

Ingredient 1

• Denote schematically by WW the water waves equation. Then for a suitable norm $\|\cdot\|$,

$$WW(M) = R, \quad \|R\| \lesssim e^{-\varepsilon_0 t} e^{-\epsilon_0 h},\ \varepsilon_0 > 0\,.$$

• The key point to get this property is the following statement.

Proposition 3

Assume that $\psi \in C_b^\infty(\mathbb{R})$ has the decay property :

$$\exists\, d > 0,\ \forall\, \alpha \in \mathbb{N}, \alpha \geq 1, \exists\, C_\alpha, \forall\, x \in \mathbb{R}, |\partial_x^\alpha \psi(x)| \leq C_\alpha e^{-d|x|}.$$

Then for $\eta \in H^\infty(\mathbb{R})$ with $H + \eta \geq c_0 > 0$, $G[\eta]\psi$ has an exponential decay, that is, for any $\alpha \geq 0$, there exist a constant C_α depending on α and $0 < \epsilon < d$ independent of α such that for every $x \in \mathbb{R}$,

$$\left|\partial_x^\alpha\Big(G[\eta]\psi\Big)(x)\right| \leq C_\alpha e^{-\epsilon|x|}.$$

• The above result uses in an essential way that the fluid domain is bounded in the z direction (we use a Poincaré inequality).

• The statement does not hold in the case of an infinite bottom (singularity at the low frequencies affects the exponential decay).

Ingredient 2 (preliminaries)

Set $\delta = e^{-\varepsilon_0 h}$. We look for the solution U under the form :

$$U = M + \delta v_1 + \delta^2 v_2 + \cdots \delta^N v_N + w\,.$$

Denote by L the linearized operator about M in the water waves equation. Then the equation for U can be reformulated as

$$\begin{aligned}
(\partial_t + L) v_1 &= \delta^{-1} R, \\
(\partial_t + L) v_2 &= Q_2(v_1), \\
(\partial_t + L) v_3 &= Q_3(v_1, v_2), \\
&\cdots \\
(\partial_t + L) v_N &= Q_N(v_1, v_2, \ldots, v_{N-1}),
\end{aligned}$$

plus an equation for w.

Ingredient 2 (statement and construction of v_j, $j \geq 1$)

• Denote by $S(t,\tau)$ resolution operator associated to $\partial_t + L$. Then the key (and most difficult) point in the proof is there are $\|\cdot\|_X$ and $\|\cdot\|_{X_0}$ such that

$$\|S(t,\tau)(f)\|_X \lesssim h^{\frac{1}{4}} e^{\frac{\varepsilon_0}{2}|t-\tau|} \|f\|_{X^0}\,.$$

In fact, $\frac{\varepsilon_0}{2}$ can be replaced by any positive constant and $h^{\frac{1}{4}}$ is a harmless loss which probably may be avoided.

• Now, we define v_1 as

$$v_1(t) = \int_t^{\infty} S(t,\tau)(\delta^{-1}R(\tau))d\tau.$$

Then

$$\|v_1(t)\|_X \lesssim h^{\frac{1}{4}} \int_t^{\infty} e^{\frac{\varepsilon_0}{2}(\tau-t)} e^{-\varepsilon_0\tau} d\tau \lesssim h^{\frac{1}{4}} e^{-\varepsilon_0 t}\,.$$

Ingredient 2 (construction of v_j, $j \geq 1$, sequel)

• Next, we define v_2 as

$$v_2(t) = \int_t^\infty S(t,\tau)(Q_2(v_1(\tau)))d\tau.$$

Then

$$\|v_2(t)\|_X \lesssim h^{\frac{3}{4}} \int_t^\infty e^{\frac{\varepsilon_0}{2}(\tau - t)} e^{-2\varepsilon_0 \tau} d\tau \lesssim h^{\frac{3}{4}} e^{-2\varepsilon_0 t}.$$

• We now define similarly v_3, v_4, $\ldots$ and we have that

$$\|v_j(t)\|_X \lesssim h^{\frac{2j-1}{4}} e^{-j\varepsilon_0 t}, \quad j \geq 1.$$

• As a consequence the equation for w will have a strongly decaying source term.

Ingredient 3 (energy estimates and construction of w)

• Let $(T_n)_{n=1}^{\infty}$ be a sequence tending to $+\infty$. We construct solutions w_n of the equation for w on $[0, T_n]$ with $w_n(T_n) = 0$.

• w_n enjoys the following energy estimate : for a suitable $\|\cdot\|$,

$$\|w_n(t)\|^2 \lesssim \Big(C + C_N\delta^{\frac{1}{2}} + \sup_{t\le\tau\le T_n} \|w_n(\tau)\|\Big)^{100} \Big(\int_t^{T_n} \|w_n(\tau)\|^2 d\tau + C_N e^{-2(N+1)\varepsilon_0 t}\Big)$$

• We take first $N \gg 1$ and then $\delta \ll 1$ so that

$$2(N+1)\varepsilon_0 > \Big(C + C_N\delta^{\frac{1}{2}} + 1\Big)^{100}.$$

• Thus, we can rewrite the energy estimate as follows

$$\frac{d}{dt}\Big[-e^{(C+C_N\delta^{\frac{1}{2}}+1)^{100}\,t} \int_t^{T_n} \|w_n(\tau)\|^2 d\tau\Big] \lesssim e^{[(C+C_N\delta^{\frac{1}{2}}+1)^{100}-2(N+1)\varepsilon_0]t}$$

Ingredient 3 (energy estimates and construction of w, sequel)

- We integrate between t and T_n and we get

$$\int_t^{T_n} \|w_n(\tau)\|^2 d\tau \lesssim e^{-2(N+1)\varepsilon_0 t}$$

and therefore

$$\|w_n(t)\|^2 \lesssim e^{-2(N+1)\varepsilon_0 t}, \quad t \in [0, T_n]$$

- We conclude by a compactness argument. Extend $w_n(t)$ by zero for $t > T_n$ and fix $\psi \in C_0^\infty(-1/2, 1/2)$ such that $\psi = 1$ on $(-1/4, 1/4)$. Then we apply a classical compactness argument to the sequence

$$\Big(\psi(t/T_n) w_n(t)\Big)_{n \geq 1}, \quad t \geq 0.$$

- The limit object is w and the multi-soliton is constructed.

On the estimate for $S(t, \tau)$

• The natural energy space for our problem is the space X, defined for $U = (U_1, U_2)$ as

$$\|U\|_X^2 = \|U_1\|_{H^1}^2 + \|(1 - \partial_x^2)^{-\frac{1}{4}} \partial_x U_2\|_{L^2}^2 .$$

• Denote by Λ the linearization of the energy E about a solitary wave Q_c. Then, according to the general Grillakis-Shatah-Strauss scheme one may expect the estimate

$$(\Lambda U, U) \gtrsim \|U\|_X^2, \quad (U, \partial_x Q_c) = (U, J^{-1} \partial_x Q_c) = 0 \ , J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2)$$

But $(U, \partial_x Q_c)$ takes different values for $U = (0, 1)$ and $U = (0, 2)$ and in fact one may show that (2) is wrong.

• Following Mielke, we can replace $(U, \partial_x Q_c)$ by (U_1, η_c) and still get (2).

• With this estimate in hand, we get that the semigroup associated to $J\Lambda$ grows as t (like for KdV). One makes a direct sum decomposition of span$(\partial_x Q_c, J\partial_x Q_c)$ with a space E which satisfies the modified orthogonality conditions.

On the estimate for $S(t,\tau)$ (sequel)

• In order to estimate $S(t,\tau)$, we use energy functional of the type :

$$E(U(t)) = -(JLU,\, U) - c_1(\partial_x J(\chi_1 U),\, \chi_1 U) - c_2(\partial_x J(\chi_2 U),\, \chi_2 U),$$

where we recall that L is the linearized about M operator in the water waves system and χ_1 and χ_2 localize near Q_{c_1} and Q_{c_2}.

• The first term in E corresponds to the energy conservation, the second and the third one to the conservation of the momentum.

• For suitable quantities $x(t) = x(U(t))$ of Sobolev type, we get estimates of type $\dot{x}(t) \lesssim \delta(h)x(t)$, where $\delta(h) \to 0$ as h goes to infinity.

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