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Transport of gaussian measures under Hamiltonian PDE's

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based on joint works with G. Genovese, T. Gunaratnam, R. Luca, Hiro Oh, F. Planchon, Ph. Sosoe, N. Visciglia, H. Weber.

The Sobolev spaces on the circle

• Let $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. We denote by $H^s(\mathbb{T})$ the Sobolev spaces on the circle.

• If

$$u(x) = \sum_{n \in \mathbb{Z}} e^{inx}\, \hat{u}(n),$$

where

$$\hat{u}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-inx}\, u(x)dx \in \mathbb{C}$$

then

$$\|u\|_{H^s}^2 := \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\hat{u}(n)|^2,$$

where

$$\langle n \rangle := (1 + n^2)^{\frac{1}{2}}.$$

• The norm H^s is induced from a natural scalar product which makes $H^s(\mathbb{T})$ a Hilbert space.

The gaussian measure μ_s

• We wish to define a gaussian measure of the form

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as a measure on a suitable functional space.

• Formally

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du = Z^{-1} \exp\Big(- \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\widehat{u}(n)|^2 \Big) \prod_{n\in\mathbb{Z}} d\,\widehat{u}(n)$$

and the last expression makes think about the well defined object

$$\prod_{n\in\mathbb{Z}} Z_n^{-1} \exp\Big(- \langle n\rangle^{2s} |\widehat{u}(n)|^2 \Big) d\,\widehat{u}(n),$$

where we formally wrote

$$Z^{-1} = \prod_{n\in\mathbb{Z}} Z_n{}^{-1} \quad (Z_n = \pi \langle n\rangle^{-2s}).$$

The gaussian measure μ_s (sequel)

- Therefore, we can define the measure μ_s

$$Z^{-1}\, e^{-\|u\|_{H^s}^2}\, du$$

as the image measure by the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}} e^{inx}\, \frac{g_n(\omega)}{\langle n\rangle^s}\,,$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ are i.i.d. complex gaussian random variables with mean 0 and variances 1, on a probability space $(\Omega, \mathcal{F}, p)$.

- **Question :** μ_s is a measure on which space ?

The gaussian measure μ_s (sequel)

- We can write for $N < M$

$$\Big\| \sum_{N \leq |n| \leq M} e^{inx} \, \frac{g_n(\omega)}{\langle n \rangle^s} \Big\|^2_{L^2(\Omega; H^\sigma(\mathbb{T}))} \simeq \sum_{N \leq |n| \leq M} \frac{\langle n \rangle^{2\sigma}}{\langle n \rangle^{2s}}$$

which tends to zero as $N \to \infty$, provided

$$\sigma < s - \frac{1}{2} \, .$$

- Therefore

$$\sum_{n \in \mathbb{Z}} e^{inx} \, \frac{g_n(\omega)}{\langle n \rangle^s} \in L^2(\Omega; H^\sigma(\mathbb{T})) \, .$$

The gaussian measure μ_s (sequel)

- We conclude that the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}} e^{inx} \frac{g_n(\omega)}{\langle n\rangle^s}$$

defines a probability measure on $H^\sigma(\mathbb{T})$, $\sigma < s - \frac{1}{2}$. In addition

$$\mu_s(H^{s-\frac{1}{2}}(\mathbb{T})) = 0\,.$$

- In particular

$$\mu_s(H^s(\mathbb{T})) = 0\,.$$

- In this constriction $H^s(\mathbb{T})$ is canonical but $H^\sigma(\mathbb{T})$ is not, it may be replaced for instance by $W^{\sigma,\infty}(\mathbb{T})$.

The Cameron-Martin theorem

• **Question :** How behaves μ_s under transformations ?

Theorem 1 (Cameron-Martin 1944)
Let $f \in H^\sigma(\mathbb{T})$ and let μ_f be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by

$$u \longmapsto f + u\,.$$

Then μ_f is absolutely continuous with respect to μ_s if and only if

$$f \in H^s(\mathbb{T}).$$

• Recalling that formally

$$d\mu_s(u) = Z^{-1}\, e^{-\|u\|^2_{H^s}}\, du$$

we may expect that

$$\frac{d\mu_f(u)}{d\mu_s(u)} = e^{-\|f\|^2_{H^s} - 2(u,f)_s}\,,$$

where $(\cdot,\cdot)_s$ stands for the H^s scalar product.

Proof of the Cameron-Martin theorem for μ_s

• Let $f \in H^s(\mathbb{T})$. Since we expect that the Radon-Nykodim derivative is exp $\Big(-\|f\|^2_{H^s} - 2(u,f)_s\Big)$ the whole issue is to show that $(u,f)_s < \infty$, μ_s almost surely which is equivalent to

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^s \widehat{f}(n)\,\overline{g_n(\omega)} < \infty, \qquad \text{a.s.}$$

which directly results directly from the independence and $f \in H^s(\mathbb{T})$.

• Let now $f \notin H^s(\mathbb{T})$. Then there is $g \in H^s$ such that $(f,g)_s = \infty$. Consider the set

$$A = \{u \in H^\sigma \,:\, (g,u)_s < \infty\}.$$

We already checked that $\mu_s(A) = 1$ (replace f by g in the discussion of the first half of the slide). The image of A under our shift is the set B defined by

$$B = \{u + f, \quad u \in A\}.$$

Clearly $A \cap B = \emptyset$ and therefore $\mu_s(B) = 0$.

Thus we found a set of measure 1 which is sent by the shilt by f map to a set of measure 0. This completes the proof.

Invariance of μ_s under the free Schrödinger evolution

Proposition 2

Let $S(t) = e^{it\partial_x^2}$. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u \longmapsto S(t)(u)$. Then $\mu_s(t) = \mu_s$.

Proof. We have that

$$S(t)\Big(\sum_{n\in\mathbb{Z}} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}\Big) = \sum_{n\in\mathbb{Z}} e^{inx}\,\frac{e^{-itn^2}g_n(\omega)}{\langle n\rangle^s}$$

which has the same distribution as

$$\sum_{n\in\mathbb{Z}} e^{inx}\,\frac{g_n(\omega)}{\langle n\rangle^s}$$

because $e^{-itn^2}g_n(\omega)$ has the same distribution as $g_n(\omega)$ (invariance of complex gaussians by rotations). This completes the proof.

$\boxed{\text{A remark}}$

- For a fixed sequence $(c_n)_{n\in\mathbb{Z}}$ the free Schrödinger evolution

$$\sum_{n\in\mathbb{Z}} c_n\, e^{inx}\, e^{-itn^2}$$

may have a complicated behaviour depending on the nature of the number t (leading to interesting number theory considerations) but the statistical behaviour under μ_s is the same for each time t.

Transport of μ_s under nonlinear transformations

Question : How behaves μ_s under the flow of the nonlinear Schrödinger equation (NLS) ? Let us start by the dispersionless model :

Theorem 3

Let $s \geq 1$ be an integer. Let $\rho_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves

$$i\partial_t u = |u|^4 u, \quad u|_{t=0} = u_0 . \tag{1}$$

Then for $t \neq 0$, the measure $\rho_s(t)$ is not absolutely continuous with respect to μ_s.

- The solution of (1) is given by

$$u(t,x) = u_0(x)\, e^{-it|u_0(x)|^4} \tag{2}$$

and the idea behind the proof is to show that a typical regularity property of the data resulting from the iterated logarithm law associated with μ_s is destroyed by the time oscillation in formula (2).

Transport of μ_s under nonlinear transformations (sequel)

But we also have :

Theorem 4

Let $s \geq 1$ be an integer. Let $\mu_s(t)$ be the image of μ_s under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u_0 \longmapsto u(t)$, where $u(t)$ solves the nonlinear Schrödinger equation

$$(i\partial_t + \partial_x^2)u = |u|^4 u, \quad u|_{t=0} = u_0\,. \tag{3}$$

Then $\mu_s(t)$ is absolutely continuous with respect to μ_s. In other words, μ_s is quasi-invariant under the flow of (3).

• We have similar results for the fractional NLS in $1d$, for the nonlinear wave equations in dimensions ≤ 3, for the gKdV equation and for BBM type models.

• Depending on the equation, we have more or less informations on the resulting Radon-Nykodim derivatives.

• I am very interested in the extension to the $2d$ NLS which seems to require some new ingredients. Even the $3d$ NLS does not seem completely out of reach ...

A corollary (L^1 stability for the corresponding Liouville equation)

Theorem 5

Let $s \geq 1$ be an integer. Let $f_1, f_2 \in L^1(d\mu_s)$ and let $\Phi(t)$ be the flow of

$$(i\partial_t + \partial_x^2)u = |u|^4 u, \quad u|_{t=0} = u_0\,,$$

defined μ_s a.s. Then for every $t \in \mathbb{R}$, the transports of the measures

$$f_1(u)d\mu_s(u), \quad f_2(u)d\mu_s(u)$$

by $\Phi(t)$ are given by

$$F_1(t,u)d\mu_s(u), \quad F_2(t,u)d\mu_s(u)$$

respectively, for suitable $F_1(t,\cdot), F_2(t,\cdot) \in L^1(d\mu_s)$. Moreover

$$\|F_1(t) - F_2(t)\|_{L^1(d\mu_s)} = \|f_1 - f_2\|_{L^1(d\mu_s)}\,.$$

• Local in time bounds for other distances are obtained in a recent work by work by Forlano-Seong. There are many further things to be understood.

Remarks

• The above results are restricted to relatively regular solutions of the equation (cf. the assumption $s \geq 1$) because the question of quasi-invariance seems *strictly more complicated* than the question of proving the existence of the dynamics (this seems to be an infinite dimensional phenomenon).

• For exemple, in the context of the impressive recent results by Deng-Nahmod-Yue for NLS with low regularity gaussian data, the question of the propagation of the gaussianity by the flow of the equation seems completely open.

• A similar remark applies to the earlier probabilistic well-posedness results by Nicolas Burq and myself on the nonlinear wave equation and by Colliander-Oh on the 1d NLS.

• I however expect that the methods and the ideas developed in the work on probabilistic well-posedness may become useful in quasi-invariance questions. Ideally, one day we will may be succeed to have a quasi-invariance result for a deterministically ill-posed posed problem.

Methods

• Roughly speaking, presently, we have two different methods to prove this kind of quasi-invariance results :

• **Method 1 :** Using the *time oscillations* (dispersive estimates).

• **Method 2 :** Using the *random oscillations* (concentration of measure estimates).

• In both methods, we do not study directly the evolution of the gaussian measure μ_s but the evolution of ρ_s defined by

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where $R_s(u)$ is a suitable correction and where χ is a continuous function with a compact support and where $H(u)$ is the Hamiltonian of the equation under consideration (conserved by the flow). We formally have

$$e^{-R_s(u)} d\mu_s(u) = Z^{-1} e^{-R_s(u)} e^{-\|u\|^2_{H^s}} du = Z^{-1} e^{-E_s(u)} du\,,$$

where

$$E_s(u) = \|u\|^2_{H^s} + R_s(u)\,.$$

Methods (sequel)

• The correction $R_s(u)$ in the energy functional

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)$$

is of fundamental importance and there are different intuitions behind its construction : normal form reductions, traces of complete integrability, modulated energies, ...

• Interestingly, in some cases the construction of $R_s(u)$ requires renormalisation arguments (as we saw in the talk by F. Otto yesterday).

• However, an important feature is that we *do not renormalise the equation which stays always the same*. Instead, we consider renormalised functionals associated with the equation with data distributed according to a gaussian field.

On method 1

- Let $\Phi(t)$ be the flow of the PDE under consideration.
- Formally the transported measure is given by

$$Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, du = \\ Z^{-1}\chi(H(u))\, e^{-E_s(\Phi(t)(u))}\, e^{E_s(u)} e^{-E_s(u)} du$$

which can be interpreted as the (relatively) well defined object

$$e^{-\Big(E_s(\Phi(t)(u))-E_s(u)\Big)}\, \chi(H(u)) e^{-R_s(u)} d\mu_s(u)\,.$$

- Therefore we hope that the Radon-Nykodim derivative of the transport of ρ_s is given by

$$e^{-\Big(E_s(\Phi(t)(u))-E_s(u)\Big)}$$

- **Problem :** In $E_s(\Phi(t)(u)) - E_s(u)$ both terms are strongly diverging on the support of μ_s but the hope is to find some cancellations thanks to PDE smoothing estimates.

On method 1 (sequel)

• More precisely, one can write

$$E_s(\Phi(t)(u)) - E_s(u) = \int_0^t \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau} d\tau.$$

Set

$$G_s(\tau) = \frac{d}{dt} E_s(\Phi(t)(u))\Big|_{t=\tau}.$$

We will be done, if we can prove that

$$\Big|\int_0^t G_s(\tau)d\tau\Big| \leq C_{H(u)} \|u\|^{\theta}_{H^{s-\frac{1}{2}-}},$$

for a suitable choice of $R_s(u)$ and for a suitable number θ.

• If E_s is a conserved quantity (Gibbs measures) then $G_s = 0$ and one expects an invariant measure. However, this may not be true at the level of the approximated finite dimensional models and a serious difficulty may appear (cf. works by Nahmod-Oh-Rey Bellet-Staffilani, Tz.-Visciglia, Genovese-Luca-Valeri, ...).

On method 1 (sequel)

• If $\theta < 2$ the Randon-Nykodim density is indeed given by

$$e^{-\Big(E_s(\Phi(t)(u))-E_s(u)\Big)}$$

in the sense that it is the natural limit of the corresponding (perfectly well defined) finite dimensional densities.
• If $\theta \geq 2$, we can define the Radon-Nykodim density of the transport of

$$\exp\Big(-\|u\|^m_{H^{s-\frac{1}{2}-}}\Big)\,\chi(H(u))\,e^{-R_s(u)}d\mu_s(u),$$

where $m \gg 1$ (depending on θ).
• **Remark.** It would be interesting to replace

$$\Big|\int_0^t G_s(\tau)d\tau\Big| \leq C_{H(u)}\|u\|^\theta_{H^{s-\frac{1}{2}-}},$$

with more subtle estimates.

On method 2

- Let $A \subset H^\sigma(\mathbb{T})$ be a measurable set.
- Recall that

$$d\rho_s(u) = \chi(H(u))\, e^{-R_s(u)}\, d\mu_s(u)\,,$$

where χ is a continuous function with a compact support and $H(u)$ is the Hamiltonian of the equation under consideration.

- Then

$$\frac{d}{dt}\rho_s(\Phi(t)(A))\Big|_{t=\bar{t}} = \frac{d}{dt}\rho_s(\Phi(t)(\Phi(\bar{t})(A)))\Big|_{t=0}$$

which is **formally** equal to

$$\int_{\Phi(\bar{t})(A)} \frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0}\, d\rho_s(u)$$
$$\leq \Big\|\frac{d}{dt}E_s(\Phi(t)(A))\Big|_{t=0}\Big\|_{L^p(\rho_s)}\Big(\rho_s(\Phi(\bar{t})(A))\Big)^{1-\frac{1}{p}}$$

On method 2 (sequel)

• We would be done if we show that

$$\left\| \frac{d}{dt} E_s(\Phi(t)(A)) \Big|_{t=0} \right\|_{L^p(\rho_s)} \leq Cp, \quad p \gg 1\,. \tag{4}$$

In the proof of the last inequality we only exploit the random oscillations of the initial data.

• Important observation : if we are only interested in the qualitative statement of quasi-invariance then in (4) we can suppose that A included in a bounded set of a Banach space $\mathcal{H}$ which is of full measure such that the PDE under consideration is globally well posed in $\mathcal{H}$ (existence, uniqueness and persistence of regularity).

• Let us **formally** show how we use (4) (similarly to the uniqueness for $2d$ Euler) to get the quasi-invariance. Set

$$x(t) = \rho_s(\Phi(t)(A))\,.$$

Thanks to (4) we have

$$\dot{x}(t) \leq Cp(x(t))^{1-\frac{1}{p}}$$

On method 2 (sequel)

Therefore

$$\frac{d}{dt}\Big((x(t))^{\frac{1}{p}}\Big) \leq C\,.$$

• An integration yields

$$(x(t))^{\frac{1}{p}} - (x(0))^{\frac{1}{p}} \leq Ct$$

Therefore, if $x(0) = 0$ then

$$x(t) \leq (Ct)^p$$

which goes to zero as $p \to \infty$, provided $Ct < 1$.
• Since the constant C is uniform we can iterate the last argument and achieve any time.
• The above argument may become rigorous if we use some approximation arguments resulting from the Cauchy problem theory of the equation under consideration.

$\boxed{\text{A final remark}}$

- Method 2 performs better for equations with weak dispersion.

- It would be interesting to find a way to combine Method 1 and Method 2 ...

Thank you for your attention !