Transport of gaussian measures under Hamiltonian PDE’s

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The Sobolev spaces on the circle

• Let $\mathbb{T} := \mathbb{R}/2\pi \mathbb{Z}$. We denote by $H^s(\mathbb{T})$ the Sobolev spaces on the circle.

• If

$$u(x) = \sum_{n \in \mathbb{Z}} e^{inx} \hat{u}(n),$$

where

$$\hat{u}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-inx} u(x) dx \in \mathbb{C},$$

then

$$\|u\|_{H^s}^2 := \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\hat{u}(n)|^2,$$

where

$$\langle n \rangle := (1 + n^2)^{\frac{1}{2}}.$$

• The norm $H^s$ is induced from a natural scalar product which makes $H^s(\mathbb{T})$ a Hilbert space.
The gaussian measure $\mu_s$

- We wish to define a gaussian measure of the form
  \[ Z^{-1} e^{-\|u\|_{H^s}^2} du \]
as a measure on a suitable functional space.
- Formally
  \[ Z^{-1} e^{-\|u\|_{H^s}^2} du = Z^{-1} \exp \left( - \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\hat{u}(n)|^2 \right) \prod_{n \in \mathbb{Z}} d\hat{u}(n) \]
  and the last expression makes think about the well defined object
  \[ \prod_{n \in \mathbb{Z}} Z_n^{-1} \exp \left( - \langle n \rangle^{2s} |\hat{u}(n)|^2 \right) d\hat{u}(n), \]
  where we formally wrote
  \[ Z^{-1} = \prod_{n \in \mathbb{Z}} Z_n^{-1} \quad (Z_n = \pi \langle n \rangle^{-2s}). \]
Therefore, we can define the measure $\mu_s$

$$Z^{-1} e^{-\|u\|_H^2} \, du$$

as the image measure by the map

$$\omega \mapsto \sum_{n \in \mathbb{Z}} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s},$$

where $(g_n(\omega))_{n \in \mathbb{Z}}$ are i.i.d. complex gaussian random variables with mean 0 and variances 1, on a probability space $(\Omega, \mathcal{F}, p)$.

**Question**: $\mu_s$ is a measure on which space?
The gaussian measure $\mu_\sigma$ (sequel)

- We can write for $N < M$

$$\left\| \sum_{N \leq |n| \leq M} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s} \right\|_{L^2(\Omega; H^\sigma(\mathbb{T}))}^2 \approx \sum_{N \leq |n| \leq M} \frac{\langle n \rangle^{2\sigma}}{\langle n \rangle^{2s}}$$

which tends to zero as $N \to \infty$, provided

$$\sigma < s - \frac{1}{2}.$$ 

- Therefore

$$\sum_{n \in \mathbb{Z}} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s} \in L^2(\Omega; H^\sigma(\mathbb{T})).$$
The gaussian measure \( \mu_s \) (sequel)

- We conclude that the map

\[
\omega \mapsto \sum_{n \in \mathbb{Z}} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s}
\]

defines a probability measure on \( H^\sigma(\mathbb{T}) \), \( \sigma < s - \frac{1}{2} \). In addition

\[
\mu_s(H^{s-\frac{1}{2}}(\mathbb{T})) = 0.
\]

- In particular

\[
\mu_s(H^s(\mathbb{T})) = 0.
\]

- In this constriction \( H^s(\mathbb{T}) \) is canonical but \( H^\sigma(\mathbb{T}) \) is not, it may be replaced for instance by \( W^{\sigma,\infty}(\mathbb{T}) \).
The Cameron-Martín theorem

- **Question**: How behaves $\mu_s$ under transformations?

**Theorem 1 (Cameron-Martín 1944)**

Let $f \in H^\sigma(T)$ and let $\mu_f$ be the image of $\mu_s$ under the map from $H^\sigma(T)$ to $H^\sigma(T)$ defined by

$$u \mapsto f + u.$$ 

Then $\mu_f$ is absolutely continuous with respect to $\mu_s$ if and only if

$$f \in H^s(T).$$

- Recalling that formally

$$d\mu_s(u) = Z^{-1} e^{-\|u\|_{H^s}^2} \, du$$

we may expect that

$$\frac{d\mu_f(u)}{d\mu_s(u)} = e^{-\|f\|_{H^s}^2 - 2(u,f)_s},$$

where $(\cdot, \cdot)_s$ stands for the $H^s$ scalar product.
Proof of the Cameron-Martin theorem for $\mu_s$

• Let $f \in H^s(\mathbb{T})$. Since we expect that the Radon-Nykodim derivative is $\exp \left( -\|f\|_{H^s}^2 - 2(u, f)_s \right)$ the whole issue is to show that $(u, f)_s < \infty$, $\mu_s$ almost surely which is equivalent to

$$\sum_{n \in \mathbb{Z}} \langle n \rangle^s \hat{f}(n) \overline{g_n(\omega)} < \infty, \quad \text{a.s.}$$

which directly results directly from the independence and $f \in H^s(\mathbb{T})$.

• Let now $f \notin H^s(\mathbb{T})$. Then there is $g \in H^s$ such that $(f, g)_s = \infty$. Consider the set

$$A = \{ u \in H^\sigma : (g, u)_s < \infty \}.$$

We already checked that $\mu_s(A) = 1$ (replace $f$ by $g$ in the discussion of the first half of the slide). The image of $A$ under our shift is the set $B$ defined by

$$B = \{ u + f, \quad u \in A \}.$$

Clearly $A \cap B = \emptyset$ and therefore $\mu_s(B) = 0$.

Thus we found a set of measure 1 which is sent by the shift by $f$ map to a set of measure 0. This completes the proof.
Invariance of $\mu_s$ under the free Schrödinger evolution

**Proposition 2**

Let $S(t) = e^{it\partial_x^2}$. Let $\mu_s(t)$ be the image of $\mu_s$ under the map from $H^\sigma(\mathbb{T})$ to $H^\sigma(\mathbb{T})$ defined by $u \mapsto S(t)(u)$. Then $\mu_s(t) = \mu_s$.

**Proof.** We have that

$$S(t) \left( \sum_{n \in \mathbb{Z}} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s} \right) = \sum_{n \in \mathbb{Z}} e^{inx} \frac{e^{-itn^2}g_n(\omega)}{\langle n \rangle^s}$$

which has the same distribution as

$$\sum_{n \in \mathbb{Z}} e^{inx} \frac{g_n(\omega)}{\langle n \rangle^s}$$

because $e^{-itn^2}g_n(\omega)$ has the same distribution as $g_n(\omega)$ (invariance of complex gaussians by rotations). This completes the proof.
A remark

For a fixed sequence \((c_n)_{n \in \mathbb{Z}}\) the free Schrödinger evolution

\[
\sum_{n \in \mathbb{Z}} c_n e^{inx} e^{-itn^2}
\]

may have a complicated behaviour depending on the nature of the number \(t\) (leading to interesting number theory considerations) but the statistical behaviour under \(\mu_s\) is the same for each time \(t\).
Question : How behaves $\mu_s$ under the flow of the nonlinear Schrödinger equation (NLS)? Let us start by the dispersionless model:

**Theorem 3**

Let $s \geq 1$ be an integer. Let $\rho_s(t)$ be the image of $\mu_s$ under the map from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$ defined by $u_0 \mapsto u(t)$, where $u(t)$ solves

$$i\partial_t u = |u|^4 u, \quad u|_{t=0} = u_0.$$  \hspace{1cm} (1)

Then for $t \neq 0$, the measure $\rho_s(t)$ is not absolutely continuous with respect to $\mu_s$.

- The solution of (1) is given by

$$u(t,x) = u_0(x) e^{-it|u_0(x)|^4}$$  \hspace{1cm} (2)

and the idea behind the proof is to show that a typical regularity property of the data resulting from the iterated logarithm law associated with $\mu_s$ is destroyed by the time oscillation in formula (2).
But we also have:

**Theorem 4**

Let \( s \geq 1 \) be an integer. Let \( \mu_s(t) \) be the image of \( \mu_s \) under the map from \( H^\sigma(\mathbb{T}) \) to \( H^\sigma(\mathbb{T}) \) defined by \( u_0 \mapsto u(t) \), where \( u(t) \) solves the nonlinear Schrödinger equation

\[
(i \partial_t + \partial_x^2)u = |u|^4u, \quad u|_{t=0} = u_0.
\]  

(3)

Then \( \mu_s(t) \) is absolutely continuous with respect to \( \mu_s \). In other words, \( \mu_s \) is quasi-invariant under the flow of (3).

- We have similar results for the fractional NLS in 1d, for the nonlinear wave equations in dimensions \( \leq 3 \), for the gKdV equation and for BBM type models.
- Depending on the equation, we have more or less informations on the resulting Radon-Nykodim derivatives.
- I am very interested in the extension to the 2d NLS which seems to require some new ingredients. Even the 3d NLS does not seem completely out of reach ...
A corollary ($L^1$ stability for the corresponding Liouville equation)

**Theorem 5**

Let $s \geq 1$ be an integer. Let $f_1, f_2 \in L^1(d\mu_s)$ and let $\Phi(t)$ be the flow of

$$(i\partial_t + \partial^2_x)u = |u|^4u, \quad u|_{t=0} = u_0,$$

defined $\mu_s$ a.s. Then for every $t \in \mathbb{R}$, the transports of the measures

$$f_1(u)d\mu_s(u), \quad f_2(u)d\mu_s(u)$$

by $\Phi(t)$ are given by

$$F_1(t,u)d\mu_s(u), \quad F_2(t,u)d\mu_s(u)$$

respectively, for suitable $F_1(t, \cdot), F_2(t, \cdot) \in L^1(d\mu_s)$. Moreover

$$\|F_1(t) - F_2(t)\|_{L^1(d\mu_s)} = \|f_1 - f_2\|_{L^1(d\mu_s)}.$$

• Local in time bounds for other distances are obtained in a recent work by work by Forlano-Seong. There are many further things to be understood.
Remarks

- The above results are restricted to relatively regular solutions of the equation (cf. the assumption $s \geq 1$) because the question of quasi-invariance seems *strictly more complicated* than the question of proving the existence of the dynamics (this seems to be an infinite dimensional phenomenon).
- For example, in the context of the impressive recent results by Deng-Nahmod-Yue for NLS with low regularity gaussian data, the question of the propagation of the gaussianity by the flow of the equation seems completely open.
- A similar remark applies to the earlier probabilistic well-posedness results by Nicolas Burq and myself on the nonlinear wave equation and by Colliander-Oh on the 1d NLS.
- I however expect that the methods and the ideas developed in the work on probabilistic well-posedness may become useful in quasi-invariance questions. Ideally, one day we will may be succeed to have a quasi-invariance result for a deterministically ill-posed posed problem.
• Roughly speaking, presently, we have two different methods to prove this kind of quasi-invariance results:
  • **Method 1**: Using the *time oscillations* (dispersive estimates).
  • **Method 2**: Using the *random oscillations* (concentration of measure estimates).
• In both methods, we do not study directly the evolution of the gaussian measure $\mu_s$ but the evolution of $\rho_s$ defined by
  \[
  d\rho_s(u) = \chi(H(u)) e^{-R_s(u)} d\mu_s(u),
  \]
  where $R_s(u)$ is a suitable correction and where $\chi$ is a continuous function with a compact support and where $H(u)$ is the Hamiltonian of the equation under consideration (conserved by the flow). We formally have
  \[
  e^{-R_s(u)} d\mu_s(u) = Z^{-1} e^{-R_s(u)} e^{-\|u\|^2_{H^s}} du = Z^{-1} e^{-E_s(u)} du,
  \]
  where
  \[
  E_s(u) = \|u\|^2_{H^s} + R_s(u).
  \]
• The correction $R_s(u)$ in the energy functional

$$E_s(u) = \|u\|_{H^s}^2 + R_s(u)$$

is of fundamental importance and there are different intuitions behind its construction: normal form reductions, traces of complete integrability, modulated energies, ...

• Interestingly, in some cases the construction of $R_s(u)$ requires renormalisation arguments (as we saw in the talk by F. Otto yesterday).

• However, an important feature is that we do not renormalise the equation which stays always the same. Instead, we consider renormalised functionals associated with the equation with data distributed according to a gaussian field.
On method 1

- Let \( \Phi(t) \) be the flow of the PDE under consideration.
- Formally the transported measure is given by

\[
Z^{-1} \chi(H(u)) e^{-E_s(\Phi(t)(u))} du = \int Z^{-1} \chi(H(u)) e^{-E_s(\Phi(t)(u))} e^{E_s(u)} e^{-E_s(u)} du
\]

which can be interpreted as the (relatively) well defined object

\[
e^{-\left(E_s(\Phi(t)(u)) - E_s(u)\right)} \chi(H(u)) e^{-R_s(u)} d\mu_s(u).
\]

- Therefore we hope that the Radon-Nykodim derivative of the transport of \( \rho_s \) is given by

\[
e^{-\left(E_s(\Phi(t)(u)) - E_s(u)\right)}
\]

- **Problem**: In \( E_s(\Phi(t)(u)) - E_s(u) \) both terms are strongly diverging on the support of \( \mu_s \) but the hope is to find some cancellations thanks to PDE smoothing estimates.
On method 1 (sequel)

- More precisely, one can write

$$E_s(\Phi(t)(u)) - E_s(u) = \int_{0}^{t} \frac{d}{dt} E_s(\Phi(t)(u)) \bigg|_{t=\tau} d\tau.$$  

Set

$$G_s(\tau) = \frac{d}{dt} E_s(\Phi(t)(u)) \bigg|_{t=\tau}.$$  

We will be done, if we can prove that

$$\left| \int_{0}^{t} G_s(\tau) d\tau \right| \leq C_{H(u)} \|u\|^{\theta}_{H^{s-\frac{1}{2}}},$$

for a suitable choice of $R_s(u)$ and for a suitable number $\theta$.

- If $E_s$ is a conserved quantity (Gibbs measures) then $G_s = 0$ and one expects an invariant measure. However, this may not be true at the level of the approximated finite dimensional models and a serious difficulty may appear (cf. works by Nahmod-Oh-Rey-Bellet-Staffilani, Tz.-Visciglia, Genovese-Luca-Valeri, ...).
On method 1 (sequel)

- If $\theta < 2$ the Randon-Nykodim density is indeed given by
  \[ e^{-\left( E_s(\Phi(t)(u)) - E_s(u) \right)} \]
  in the sense that it is the natural limit of the corresponding (perfectly well defined) finite dimensional densities.

- If $\theta \geq 2$, we can define the Radon-Nykodim density of the transport of
  \[ \exp \left( -\|u\|_{H^{s-\frac{1}{2}}}^m \right) \chi(H(u)) e^{-R_s(u)} d\mu_s(u), \]
  where $m \gg 1$ (depending on $\theta$).

- **Remark.** It would be interesting to replace
  \[ \left| \int_0^t G_s(\tau)d\tau \right| \leq C_H(u)\|u\|_{H^{s-\frac{1}{2}}}^{\theta}, \]
  with more subtle estimates.
On method 2

• Let \( A \subset H^\sigma(\mathbb{T}) \) be a measurable set.
• Recall that

\[
d\rho_s(u) = \chi(H(u)) e^{-R_s(u)} d\mu_s(u),
\]

where \( \chi \) is a continuous function with a compact support and \( H(u) \) is the Hamiltonian of the equation under consideration.
• Then

\[
\left. \frac{d}{dt} \rho_s(\Phi(t)(A)) \right|_{t=\bar{t}} = \left. \frac{d}{dt} \rho_s(\Phi(t)(\Phi(\bar{t})(A))) \right|_{t=0}
\]

which is formally equal to

\[
\int_{\Phi(\bar{t})(A)} \frac{d}{dt} E_s(\Phi(t)(A)) \bigg|_{t=0} d\rho_s(u)
\]

\[
\leq \left\| \frac{d}{dt} E_s(\Phi(t)(A)) \bigg|_{t=0} \right\|_{L^p(\rho_s)} \left( \rho_s(\Phi(\bar{t})(A)) \right)^{1-\frac{1}{p}}
\]
On method 2 (sequel)

- We would be done if we show that
  \[ \left\| \frac{d}{dt} E_s(\Phi(t)(A)) \right\|_{t=0}^{L^p(\rho_s)} \leq C_p, \quad p \gg 1. \]  \hspace{1cm} (4)

In the proof of the last inequality we only exploit the random oscillations of the initial data.

- Important observation: if we are only interested in the qualitative statement of quasi-invariance then in (4) we can suppose that \( A \) included in a bounded set of a Banach space \( \mathcal{H} \) which is of full measure such that the PDE under consideration is globally well posed in \( \mathcal{H} \) (existence, uniqueness and persistence of regularity).

- Let us formally show how we use (4) (similarly to the uniqueness for 2d Euler) to get the quasi-invariance. Set

  \[ x(t) = \rho_s(\Phi(t)(A)). \]

Thanks to (4) we have

\[ \dot{x}(t) \leq C_p(x(t))^{1-\frac{1}{p}} \]
Therefore

\[
\frac{d}{dt}\left((x(t))^{\frac{1}{p}}\right) \leq C.
\]

- An integration yields

\[
(x(t))^{\frac{1}{p}} - (x(0))^{\frac{1}{p}} \leq Ct
\]

Therefore, if \(x(0) = 0\) then

\[
x(t) \leq (Ct)^p
\]

which goes to zero as \(p \to \infty\), provided \(Ct < 1\).
- Since the constant \(C\) is uniform we can iterate the last argument and achieve any time.
- The above argument may become rigorous if we use some approximation arguments resulting from the Cauchy problem theory of the equation under consideration.
• Method 2 performs better for equations with weak dispersion.

• It would be interesting to find a way to combine Method 1 and Method 2 ...
Thank you for your attention!