Transverse stability issues in Hamiltonian PDE

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Abstract. We present results concerning the transverse stability of one dimensional solitary waves subject to periodic transverse perturbations in the context of the KP equations and the water-waves system.

1 Stability of solitary waves for the KdV and the KP equations

The Korteweg–de Vries (KdV) equation and the Kadomtsev-Petviashvili (KP) equations are derived as asymptotic models (see, e.g., [12]) from the much more complicated, but derived from first principles, water-waves system (the water-waves system will be presented below). The KdV and the KP equations have a remarkably deep structure. We believe that it is worth to always keep in mind that they are derived from the water-waves system and therefore to try to understand which of their properties are still true for the water-waves system. Below will be guided by this philosophy and we will try to focus on those properties of the KdV and the KP equations which may have at least partial analogues at the level of the water-waves system. As we will see below some of the properties are in fact already extended at the level of the water-waves system, others are challenging open problems.

The KdV equation reads

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \tag{1.1}$$

where the unknown u is a real valued function. The KdV equation has a well-known particular solution

$$S_c(t,x) = cQ\big(\sqrt{c}(x-ct)\big), \quad c>0, \quad Q(x) = 3\mathrm{ch}^{-2}(x/2) \tag{1.2}$$

called a solitary wave. In (1.2) the positive constant c represents the propagation speed and one may think of (1.2) as the displacement of the graph of the function $S_c(0,x)$ from left to the right with constant speed c.

The orbital stability of the KdV solitary wave $S_c(t,x)$ was first studied by Benjamin in [3]. Thanks to the work of Kenig-Ponce-Vega [10] we know that the KdV equation (1.1) is globally well-posed in the Sobolev space $H^1(\mathbb{R})$ and combining this fact with the Benjamin analysis leads to the following statement.

Theorem 1.1 *The solitary wave $S_c(t,x)$ is (orbitally) stable as a solution of the KdV equation. More precisely, for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $u_0 \in H^1(\mathbb{R})$ such that*

$$\|u_0(x) - S_c(0,x)\|_{H^1} < \delta$$

the solution of the KdV equation with initial datum u_0 satisfies

$$\sup_{t\in\mathbb{R}} \inf_{a\in\mathbb{R}} \|u(t,x-a) - S_c(t,x)\|_{H^1} < \varepsilon. \tag{1.3}$$

The translations in (1.3) are needed. Indeed, one cannot have

$$\sup_{t\in\mathbb{R}} \|u(t,x) - S_c(t,x)\|_{H^1} < \varepsilon$$

as can be seen by considering an initial data of the form $S_{\tilde{c}}(0,x)$ with $\tilde{c}$ close to c.

The choice of the Sobolev space $H^1(\mathbb{R})$ as a distance to measure the stability phenomenon is natural when having in mind the conservation laws for the KdV equation (1.1). One easily verifies that the L^2 norm and the energy defined as

$$E(u) = \int_{\mathbb{R}} (\partial_x u)^2 - \frac{1}{3}\int_{\mathbb{R}} u^3$$

are conserved by the flow of (1.1). As a consequence, using the Sobolev inequality one may identify $H^1(\mathbb{R})$ as the set of the functions with bounded energy and L^2 norm. These two conservation laws play a key role in the proof of Theorem 1.1. As we shall see below one can also prove asymptotic stability results for the solitary waves of (1.1).

When studying the stability of the KdV solitary waves under transverse perturbations, the Soviet physicists Kadomtsev and Petviashvily introduced in [8] the two dimensional models

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u) \pm \partial_y^2 u = 0 \tag{1.4}$$

called KP-I and KP-II equations depending on the sign in front of $\partial_y^2 u$ (the sign plus gives KP-II while the sign minus gives KP-I). As mentioned above the KP equations can also be obtained from the water-waves system (see [12]). Clearly the KdV solitary wave $S_c(t,x)$ solves (1.4) as well. The *remarkable formal analysis* in [8] leads to the believe that the KdV solitary wave is stable as a solution of the KP-II equation and unstable as a solution of the KP-I equation. However a mathematically rigorous proof of such statements was out of reach at the time of the writing of [8] for several reasons. Among the many issues to be resolved an important point is to define a suitable analytic framework where one can prove that the KP equations (1.4) have a well-defined dynamics, at least close to the solitary waves. The natural idea we adopted in the works [6, 17, 19, 23, 27] for an analytic framework in the studying of (1.4) was to consider (1.4) posed on the the product space $\mathbb{R}\times\mathbb{T}$, i.e., for $x \in \mathbb{R}$ and $y \in \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ denotes a one dimensional torus. In other words, we shall consider solutions of (1.4) which are localised in x (as $S_c(t,x)$ is) and periodic in the transverse variable y with period 2π. The choice of 2π is of course not canonical and any other period can be considered as well. However by using the scale invariance[1] of the KP equations, we can always reduce the matters to the period 2π. The KdV solitary wave $S_c(t,x)$ is seen as a solution of (1.4), periodic in y (with any period). Therefore, we will be in the situation when *we study the stability of the KdV solitary waves as solutions of the KP equations, subject to periodic transverse perturbations.*

Let us first consider the KP-I equation, posed $\mathbb{R}\times\mathbb{T}$,

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u) - \partial_y^2 u = 0, \quad x \in \mathbb{R},\ y \in \mathbb{T}. \tag{1.5}$$

The L^2 norm is (at least formally) conserved by the flow of (1.5) . So is the energy

$$E(u) = \int_{\mathbb{R}\times\mathbb{T}} (\partial_x u)^2 + \int_{\mathbb{R}\times\mathbb{T}} (\partial_x^{-1}\partial_y u)^2 - \frac{1}{3}\int_{\mathbb{R}\times\mathbb{T}} u^3,$$

where ∂_x^{-1} is defined via the Fourier transform as the multiplication with the singular multiplier $(i\xi)^{-1}$. In fact there is an infinite sequence of *formal conservation laws* associated with the KP-I equation [31]. For instance, the next one after the energy is of the form

$$\int_{\mathbb{R}\times\mathbb{T}} (\partial_x^2 u)^2 + \int_{\mathbb{R}\times\mathbb{T}} (\partial_x^{-2}\partial_y^2 u)^2 + \text{l.o.t.}, \tag{1.6}$$

where by l.o.t. we mean terms which become negligible if we use the controls given by the energy and the L^2 conservation laws. As it was observed in [18], because of the presence of antiderivatives in the KP-I conservation laws, there is a serious analytic obstruction to find a framework which gives sense of the next[2] after (1.6) conservation law of the KP-I equation. Inspired by the structure of the KP-I conservation laws, we can define the spaces $Z^s = Z^s(\mathbb{R}\times\mathbb{T})$ as

$$Z^s = \{u : \|(1 + |\xi|^s + |\xi^{-1}k|^s)\hat{u}(\xi,k)\|_{L^2(\mathbb{R}_\xi\times\mathbb{Z}_k)} < \infty\}$$

and equipped with the natural norm (here by $\hat{u}(\xi,k)$ we denote the Fourier transform of functions on the product space $\mathbb{R}\times\mathbb{T}$). These spaces are natural candidates for studying the global well-posedness of the KP-I equation. The following result is due to Ionescu–Kenig.

Theorem 1.2 ([6]) *The KP-I equation* (1.5) *is globally well-posed in* $Z^2(\mathbb{R}\times\mathbb{T})$.

The result of Theorem 1.2 applies equally well for the KP-I equation posed on $\mathbb{R}\times\mathbb{T}_L$, where $T_L = \mathbb{R}/(2\pi L\mathbb{Z})$ with an obvious modification of the spaces Z^s. The proof of Theorem 1.2 is based on an application of the idea introduced by Koch and the author in [11] to study low regularity well-posedness of quasilinear dispersive PDE's, combined with the three conservation laws described above. As usual

1. If u is a solution of (1.4) then so is $u_\lambda(t,x,y) = \lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$ for every $\lambda > 0$.
2. As described in [31].

by global well-posedness we mean the existence, the uniqueness, the persistence of higher regularity and the continuous dependence with respect to the time and to the initial data. Therefore the KP-I equation becomes a well-defined dynamical system on $Z^2(\mathbb{R}\times\mathbb{T})$. We shall study the stability of the KdV solitary waves as solutions of the KP-I equation in the context of this dynamics. We first state the instability result.

Theorem 1.3 ([23]) *The KdV solitary wave $S_c(t,x)$ is orbitally unstable as a solution of the KP-I equation* (1.5), *provided $c > 4/\sqrt{3}$. More precisely, for every $s \geq 0$ there exists $\eta > 0$ such that for every $\delta > 0$ there exists $u_0^\delta \in Z^2 \cap H^s$ and a time $T^\delta \approx |\log \delta|$ such that*

$$\|u_0^\delta(x,y) - S_c(0,x)\|_{H^s(\mathbb{R}\times\mathbb{T})} + \|u_0^\delta(x,y) - S_c(0,x)\|_{Z^2(\mathbb{R}\times\mathbb{T})} < \delta$$

and the (global) solution of the KP-I equation, defined by Theorem 1.2 *with datum u_0^δ satisfies*

$$\inf_{a\in\mathbb{R}} \|u(T^\delta, x-a, y) - S_c(T^\delta, x)\|_{L^2(\mathbb{R}\times\mathbb{T})} > \eta.$$

The approach used in the proof of Theorem 1.3 has its origin in the work of Grenier [5]. The proof of Theorem 1.3 could be obtained (at least for some values of c) by using the construction of explicit solutions of the KP-I equation, based on inverse scattering methods, performed in the remarkable work by Zakharov [30]. The advantage of the approach of [23] is that it is quite flexible and can be adapted to more general *non-integrable settings*. As we shall see below the approach of [23] applies to the water-waves system (see also [24] for applications to many other dispersive models). Thanks to a reversibility property of the KP-I equation the result of Theorem 1.3 also holds for negative times.

Using the above mentioned scale invariance of the KP equations we can restate the result of Theorem 1.3 for a fixed speed but allowing only sufficiently small periods, i.e., only considering perturbations of period $2\pi L$ for L small enough (in the context of (1.5) posed on $\mathbb{R}\times\mathbb{R}/(2\pi L\mathbb{Z})$). More generally, when studying the transverse stability of the KdV solitary waves under the KP flows it is equivalent to consider fixed period perturbations and vary the speed or fixing the speed and varying the periods of the transverse perturbations (such a property however does not seem to hold for the water-waves system).

The result of Theorem 1.3 only applies for large speed solitary waves. It is therefore natural to ask what happens for smaller speed solitary waves. The following statement gives an almost complete answer to this question.

Theorem 1.4 ([27]) *The KdV solitary wave $S_c(t,x)$ is orbitally stable as a solution of the KP-I equation* (1.5), *provided $c < 4/\sqrt{3}$. More precisely, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial datum u_0 of the KP-I equation* (1.5) *satisfies $u_0 \in Z^2(\mathbb{R}\times\mathbb{T})$ and*

$$\|u_0(x,y) - S_c(0,x)\|_{Z^1(\mathbb{R}\times\mathbb{T})} < \delta$$

then the solution of the KP-I equation, defined by Theorem 1.2 *with datum u_0 satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{a\in\mathbb{R}} \|u(t,x-a,y) - S_c(t,x)\|_{Z^1(\mathbb{R}\times\mathbb{T})} < \varepsilon.$$

The study of the critical speed ($c = 4/\sqrt{3}$) solitary waves is a delicate open problem.

Let us now turn to the KP-II equation

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u) + \partial_y^2 u = 0, \quad x \in \mathbb{R},\ y \in \mathbb{T}. \tag{1.7}$$

In the case of the KP-II equation the only useful conservation law from [31] is the L^2 norm. This makes the global well-posedness problem for the KP-II equation quite difficult. The global well-posedness in L^2 of the KP-II equation, posed on $\mathbb{T}^2$ and $\mathbb{R}^2$ was obtained in the remarkable work by Bourgain [4]. It was shown in [19] that the approach of Bourgain can also be applied in the context of the KP-II equation posed on $\mathbb{R} \times \mathbb{T}$. More precisely, we have the following statement.

Theorem 1.5 ([19]) *The KP-II equation* (1.7) *is globally well-posed in* $L^2(\mathbb{R} \times \mathbb{T})$.

Therefore the KP-II equation becomes a well-defined dynamical system on $L^2(\mathbb{R} \times \mathbb{T})$ and again, we shall study the stability of the KdV solitary waves as solutions of the KP-II equation in the context of this dynamics. As predicted in [8], it turns out that the KdV solitary waves are stable as solutions of the KP-II equation *for all speeds* $c > 0$. Here is the precise statement.

Theorem 1.6 ([17]) *The KdV solitary wave* $S_c(t,x)$ *is orbitally stable as a solution of the KP-II equation* (1.7) *for all* $c > 0$*. More precisely, for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if the initial datum* u_0 *of the KP-II equation* (1.7) *satisfies*

$$\|u_0(x,y) - S_c(0,x)\|_{L^2(\mathbb{R}\times\mathbb{T})} < \delta$$

then the solution of the KP-II equation, defined by Theorem 1.5 with datum u_0 *satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{a\in\mathbb{R}} \|u(t,x-a,y) - S_c(t,x)\|_{L^2(\mathbb{R}\times\mathbb{T})} < \varepsilon.$$

Moreover, there is also an asymptotic stability in the following sense. There exists a constant $\tilde{c}$ *satisfying* $\tilde{c} - c = O(\delta)$ *and a modulation parameter* $x(t)$ *satisfying*

$$\lim_{t\to\infty} \dot{x}(t) = \tilde{c}$$

and such that

$$\lim_{t\to\infty} \|u(t,x,y) - S_{\tilde{c}}(0,x-x(t))\|_{L^2((x\geq ct/2)\times\mathbb{T}_y)} = 0.$$

The approach used in the proof of Theorem 1.6 is inspired by the work by Merle–Vega [13] where one obtains an asymptotic stability result for the KdV equation under L^2 perturbations. Let us observe that Theorem 1.6 contains the Merle-Vega result as a very particular case (when $u_0(x,y)$ is y independent). Observe that because of the lack of useful higher order conserved quantities the L^2 distance is the only one where one may expect to measure the stability phenomenon for KP-II. For the KdV equation there are higher order conservation laws providing controls on higher Sobolev norms and thus one can have stability statements in much stronger than L^2 topologies. We refer to Section 4 for further details on the proof of Theorem 1.6.

2 Extensions to the water-waves system

We are now going to discuss how much, at the present moment, the results presented in the previous section can be extended to the case of the water-waves system (which is at the origin of the derivation of the KdV and the KP models).

2.1 Solitary waves for the water-waves system

The first natural question is whether the water-waves system has solitary waves of type (1.2). The answer of this question is a priori not clear at all but it was shown in the remarkable work by Amick–Kirchgässner [2] that the full water-waves system still has one dimensional solitary waves of type (1.2). In order to present the result of [2], we introduce the water-waves system. The water-waves system describes the evolution of an irrotational fluid motion in the presence of a free surface. We suppose that the bottom is finite and flat. When we study solitary waves of speed c, after some elementary reductions, we obtain that the water-waves system reads

$$\partial_t \eta = \partial_x \eta + G[\eta]\varphi, \tag{2.1}$$

$$\partial_t \varphi = \partial_x \varphi - \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\frac{\left(G[\eta]\varphi + \nabla\varphi\cdot\nabla\eta\right)^2}{1+|\nabla\eta|^2} - \alpha\eta + \beta\nabla\cdot\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}, \tag{2.2}$$

where $\eta = \eta(t,x,y)$, $\varphi = \varphi(t,x,y)$, $t,x,y\in\mathbb{R}$, $\nabla = (\partial_x,\partial_y)$ and

$$\alpha = \frac{gh}{c^2}, \quad \beta = \frac{b}{hc^2}\,.$$

Here g is the gravity constant, b takes into account the surface tension effects, h represents the deepness of the fluid domain and $G[\eta]$ is a Dirichlet-Neumann map. By definition

$$(G[\eta]\varphi)(x,y) = \partial_z\phi(x,y,\eta(x,y)) - \nabla\eta(x,y)\cdot\nabla\phi(x,y,\eta(x,y)),$$

where (for $\|\eta\|_{L^\infty} \ll 1$) the function $\phi = \phi(x,y,z)$ is the (well-defined) solution of the elliptic boundary value problem

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\phi = 0, \quad \text{in} \quad \{(x,y,z)\in\mathbb{R}^3 : -1 < z < \eta(x,y)\},$$
$$\phi(x,y,\eta(x,y)) = \varphi(x,y), \quad \partial_z\phi(x,y,-1) = 0, \quad (x,y)\in\mathbb{R}^2.$$

One may show that the map $G[\eta]$ is a first order pseudo-differential operator with principal symbol $((1+|\nabla\eta|^2)|\xi|^2 - (\nabla\eta\cdot\xi)^2)^{\frac{1}{2}}$, $\xi\in\mathbb{R}^2$.

In the context of the water-waves system a solitary wave of speed c is an independent of t and y solution of the system (2.1)-(2.2). We now can state the result of Amick–Kirchgässner.

Theorem 2.1 ([2]) *Suppose that* $\alpha = 1+\varepsilon^2$ *and* $\beta > 1/3$*. Then there exists* ε_0 *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *there is a stationary solution* $(\eta_\varepsilon(x), \varphi_\varepsilon(x))$ *of the water-waves problem of the form*

$$\eta_\varepsilon(x) = \varepsilon^2\Theta(\varepsilon x, \varepsilon), \quad \varphi_\varepsilon(x) = \varepsilon\Phi(\varepsilon x, \varepsilon),$$

where there exists $d > 0$ *such that* Θ *and* Φ *satisfy*

$$\forall\, \alpha \geq 0, \quad \exists\, C_\alpha > 0, \quad \forall\, (x, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_0), \quad \big|(\partial_x^\alpha \Theta)(x, \varepsilon)\big| \leq C_\alpha e^{-d|x|}$$

and

$$\forall\, \alpha \geq 1, \quad \exists\, C_\alpha > 0, \quad \forall\, (x, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_0), \quad \big|(\partial_x^\alpha \Phi)(x, \varepsilon)\big| \leq C_\alpha e^{-d|x|}.$$

Observe that the solitary waves established by the above result are of speed essentially $\sqrt{gh}$.

2.2 Stability with respect to $1d$ perturbations

A large part of the result of Benjamin has an analogue in the context of the water-waves system. More preciseley, thanks to a work by Mielke [14], the solitary wave of Amick–Kirchgässner is orbitally stable by 1d perturbations, as far as the local solution exists, i.e., the stability holds under an assumption on the global well-posedness of the Cauchy problem. More precisely, under the last hypothesis, for every $\kappa > 0$ there exists $\delta > 0$ such that if the initial data is independent of y and is δ close to the solitary wave in the energy space associated with the water-waves system, then the corresponding solution is κ closed to a suitable (depending on the time) spatial translate of the solitary wave, again in the energy space.

2.3 Transverse instability of the solitary water-waves

In this section we consider (2.1)–(2.2) posed on $\mathbb{R} \times \mathbb{T}_L$, where $T_L = \mathbb{R}/(2\pi L\mathbb{Z})$, i.e. we will study solutions of the water-waves system (2.1)–(2.2) which are localised in x and periodic in y with a suitable period. It turns out that in such a functional setting the solitary-waves of Amick–Kirchgässner can be destabilised if they are perturbed by transverse perturbations with a sufficiently large period. Here is a precise statement.

Theorem 2.2 (follows from [25] and [27]) *Suppose that* $\alpha = 1 + \varepsilon^2$ *and* $\beta > 1/3$*. There exists* ε_0 *such that for every* $\varepsilon \in (0, \varepsilon_0)$ *there is* $L_0 > 0$ *such that for* $L > L_0$ *the following holds true. For every* $s \geq 0$*, there exists* $\kappa > 0$ *such that for every* $\delta > 0$*, there exist* $(\eta_0^\delta(x,y), \varphi_0^\delta(x,y))$ *and a time* $T^\delta \sim |\log\delta|$ *such that*

$$\|(\eta_0^\delta(x,y), \varphi_0^\delta(x,y)) - (\eta_\varepsilon(x), \varphi_\varepsilon(x))\|_{H^s(\mathbb{R}\times\mathbb{T}_L)\times H^s(\mathbb{R}\times\mathbb{T}_L)} \leq \delta$$

and a solution $(\eta^\delta(t,x,y), \varphi^\delta(t,x,y))$ *of the water-waves system* (2.1)–(2.2) *posed on* $\mathbb{R}\times\mathbb{T}_L$ *with initial datum* $(\eta_0^\delta, \varphi_0^\delta)$*, defined on* $[0, T^\delta]$ *and satisfying*

$$\inf_{a\in\mathbb{R}} \|(\eta^\delta(T^\delta,x,y), \varphi^\delta(T^\delta,x,y)) - (\eta_\varepsilon(x-a), \varphi_\varepsilon(x-a))\|_{L^2(\mathbb{R}\times\mathbb{T}_L)\times L^2(\mathbb{R}\times\mathbb{T}_L)} > \kappa.$$

In the case $\beta > 1/3$ (strong surface tension) the asymptotic model obtained from the water-waves system in the scaling of Theorem 2.1 is the KP-I equation. Therefore Theorem 2.2 can be seen as an analogue of Theorem 1.3 for the water-waves. The existence of the solution in Theorem 2.2 is a nontrivial part of the statement. The proof of Theorem 2.2 follows by the considerations in [25] combined with the linear stability analysis performed in [27] (recalled in details in the next section in the context of the KP-I equation). The main reason for which we succeeded to extend the proof of Theorem 1.3 to the case of the water-waves is that we found a flexible approach mainly based on the Hamiltonian structure and soft spectral properties. These spectral properties are soft enough so that we can deduce them from the corresponding properties of the KdV equation. In other words the only place in the proof of Theorem 2.2 where purely KdV properties are really used is the analysis of the linearisation of the water-waves Hamiltonian about the Amick–Kirchgässner solitary wave. The spectral analysis of this linearisation is done in a perturbative way with respect to the corresponding KdV linearisation of the Hamiltonian about the solitary wave.

The proof of Theorem 2.2 is technically quite involved and giving a more detailed presentation on it would go beyond the scope of this exposé.

3 On the proof of the results for KP-I

We first discuss the instability result of Theorem 1.3. We will choose the speed $c = 1$ and we will vary the period, i.e., we shall study (1.5), posed on $\mathbb{R} \times \mathbb{T}_L$, where $T_L = \mathbb{R}/(2\pi L\mathbb{Z})$ and the issue will be to prove the instability of $Q(x-t)$ as a solution of (1.5) for L large enough. After a change of frame $Q(x)$ becomes a stationary solution of

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u - \partial_x u) - \partial_y^2 u = 0 \tag{3.1}$$

and the issue is to study the stability of $Q(x)$ as a solution of (3.1), posed on $\mathbb{R} \times \mathbb{T}_L$. Linearising (3.1) about Q leads to

$$\partial_x(\partial_t u + \partial_x(Qu) + \partial_x^3 u - \partial_x u) - \partial_y^2 u = 0$$

which can be written in a Hamiltonian form as follows

$$\partial_t u = J\Lambda u, \quad J = \partial_x,\ \Lambda = -\partial_x^2 + 1 + \partial_x^{-2}\partial_y^2 - Q.$$

In order to detect linear instable modes, we look for solutions of the last equation of the form

$$u(t,x,y) = e^{\sigma t}e^{iky}v(x)$$

which leads to the eigenvalue problem

$$\sigma v = J\Lambda(k)v, \quad \Lambda(k) = -\partial_x^2 + 1 - k^2\partial_x^{-2} - Q. \tag{3.2}$$

The operator $\Lambda(0)$ is the linearisation about Q of the KdV Hamiltonian and its spectrum is well-understood by the Sturm-Liouville theory. It turns out that $\Lambda(0)$ has one simple negative eigenvalue, zero as a simple eigenvalue and the remaining part of the spectrum is included in $[\alpha, \infty)$ for some $\alpha > 0$. The difficulty of the eigenvalue problem (3.2) is that the operator $J\Lambda(k)$ is not symmetric and thus more difficult to analyse. We can however reduce the analyses to a symmetric operator if we look for v under the form $v = Jw$ $(= \partial_x w)$. Thus we are reduced to the eigenvalue problem

$$-\sigma Jw = M(k)w, \quad M(k) = -J(-\partial_x^2 + 1 - k^2\partial_x^{-2} - Q)J. \tag{3.3}$$

Observe that $M(k)$ is not only symmetric but it also does not contain anti-derivatives. By analysing the corresponding quadratic form we obtain that the structure of the spectrum of $M(0)$ is similar to the one concerning $\Lambda(0)$ described above. We next observe that $M(k)$ is increasing in k which makes that the spectrum of $M(k)$ shifts to the right when k is increasing. Therefore there exists a $k_0 > 0$ such that $M(k_0)$ is nonnegative and has a one-dimensional kernel (see [26] for more details). Moreover, since $M'(k)$ is positive, we get that there exists a unique $k_0 \neq 0$ such that $M(k_0)$ has a non-trivial kernel. Next, using the implicit function Theorem we have that for every σ real and close to zero, there exists $k(\sigma)$, $w(\sigma)$ depending smoothly on σ, and solutions of (3.3) such that $k(0) = k_0$ and $w(\sigma) = \chi + W(\sigma)$, $W(0) = 0$, with χ an element of the kernel of $M(k_0)$ and

$$(W(\sigma), \chi) = 0, \quad \|\chi\|_{L^2(\mathbb{R})} = 1. \tag{3.4}$$

By taking the derivative of (3.3) with respect to σ, we first obtain

$$-J\chi = k'(0)M'(k_0)\chi + M(k_0)W'(0).$$

Consequently, by taking the scalar product with χ, we get

$$k'(0) = 0, \quad M(k_0)W'(0) = -J\chi. \tag{3.5}$$

Next, we can compute the second derivative. This yields

$$-2JW'(0) = k''(0)M'(k_0)\chi + M(k_0)W''(0)$$

and hence by using (3.5), we obtain that

$$k''(0) = -2\frac{(JW'(0), \chi)}{(M'(k_0)\chi, \chi)} = 2\frac{(W'(0), J\chi)}{(M'(k_0)\chi, \chi)} = -2\frac{(M(k_0)W'(0), W'(0))}{(M'(k_0)\chi, \chi)} < 0.$$

Indeed, the numerator is positive by using that $M(k_0)$ is positive on the orthogonal of χ and that $W'(0)$ is orthogonal to χ thanks to (3.4). This proves that for σ close to zero, we have

$$k(\sigma) = k_0 - \kappa\sigma^2 + \cdots$$

with $\kappa > 0$ and hence that the instability occurs for $k < k_0$. Moreover, by using an argument of Pego–Weinstein (see [21]), since $M(k)$ has at most one negative eigenvalue, we know that there exists at most one solution of (3.3) with σ of positive real part (and thus that σ is necessarily real). Consequently we get that there exists a continuous curve $\sigma(k)$ describing solutions of (3.3) with $\sigma = \sigma(k)$ defined on a maximal interval (K^*, k_0) for k such that $\sigma(k) > 0$ for every k in this interval. We claim that $K^* = 0$. Indeed, if $K^* > 0$ since σ remains bounded (see [24]) the only possibility is that $\lim_{k\to K^*} \sigma(k) = 0$. But this implies that $M(K^*)$ has a non-trivial kernel which is a contradiction with the uniqueness of k_0. Consequently, we get that there is a nontrivial solution of (3.3) with $\sigma > 0$ for every $k \in (0, k_0)$. This in turn implies that the KP-I equation posed on $\mathbb{R} \times \mathbb{T}_L$ has linear instable modes as far as $L > 1/k_0$.

REMARK 3.1 In the case of the KP-I equation the previous reasoning can be avoided by performing the ODE analysis of [1]. This ODE analysis also determines the exact value of the critical speed (or period). The argument, we have just presented has the advantage to be very flexible and in particular it applies equally well to the water-waves system. This explains why in Theorem 2.2 we can destabilise the solitary water-waves with transverse perturbations of any sufficiently large period. The exact value of k_0 in the context of the water-waves system is however not clear to us.

We now turn to the nonlinear part of the proof of Theorem 1.3. We look for a solution of the KP-I equation

$$\partial_x(\partial_t u + u\partial_x u + \partial_x^3 u - \partial_x u) - \partial_y^2 u = 0,$$

posed on $\mathbb{R} \times \mathbb{T}_L$ under the form

$$u(t) = u_{ap}(t) + v(t), \quad t \geq 0, \tag{3.6}$$

with

$$u_{ap}(t) = \sum_{k=0}^{M} \delta^k u_k(t),$$

where $|\delta| \ll 1$, $M \gg 1$ and $u_k(t)$ are defined iteratively as we explain below. We put u_{ap} in the KP-I equation and we develop in terms of the powers of δ. Clearly u_0 should solve the KP-I equation. We set $u_0 \equiv Q$. The second term should solve $\partial_t u_1 = J\Lambda u_1$. We look for u_1 under the form

$$u_1(t) = e^{\sigma t}\varphi(x, y), \quad \sigma > 0. \tag{3.7}$$

Therefore φ should solve

$$J\Lambda\varphi = \sigma\varphi, \quad \sigma > 0. \tag{3.8}$$

We now analyse solutions of (3.8) under the form

$$\varphi(x, y) = e^{i\frac{n_0 y}{L}}\psi(x) \tag{3.9}$$

for some integer $n_0 \neq 0$ with a real valued $\psi \in \cap_s H^s(\mathbb{R})$. We already know that if $L > 1/k_0$ there is a solution with $n_0 = 1$. Moreover, it can be shown that for $|n_0| \gg 1$

there is no solution of (3.8) of the form (3.9) (see [24]). We also can show that for each n_0 there cannot be more than one σ such that (3.8)–(3.9) hold. Therefore there is n_0 and σ such that (3.8)–(3.9) holds and σ is the largest possible (there is a finite number of choices for σ and thus there is a maximal one). We call σ_0 the maximal σ and we define u_1 as

$$u_1(t) \equiv e^{\sigma_0 t} e^{i\frac{n_0 y}{L}} \psi(x) + e^{\sigma_0 t} e^{-i\frac{n_0 y}{L}} \psi(x),$$

where n_0 is the value of the corresponding transverse frequency. Moreover, thanks to [1], we know that $\psi(x) = \partial_x^2 V(x)$, where $V \in \cap_s H^s(\mathbb{R})$. This is of importance in order to get that the initial perturbation belongs to Z^2.

Next, for $k \geq 2$, $u_k(t)$ is defined as a solution of the linear problem

$$\partial_t u_k - J\Lambda u_k + \frac{1}{2}\partial_x \left(\sum_{j=1}^{k-1} u_j u_{k-j} \right) = 0, \quad u_k(0) = 0.$$

Using some *delicate semi-group estimates*, we obtain the natural bounds

$$\|u_k(t)\|_{H^s(\mathbb{R}\times\mathbb{R}/(2\pi L\mathbb{Z}))} \leq C_{k,s} e^{k\sigma_0 t}. \tag{3.10}$$

Now, we set

$$T^\delta \equiv \frac{\log(\kappa/\delta)}{\sigma_0},$$

where $\kappa \ll 1$ is a small positive parameter, independent of δ, to be chosen in the sequel. As a consequence of (3.10), we get

$$||R(t)||_{H^s(\mathbb{R}\times\mathbb{R}/(2\pi L\mathbb{Z}))} \leq C_{M,s}\delta^{M+1} e^{(M+1)\sigma_0 t}, \quad t \in [0, T^\delta], \tag{3.11}$$

where

$$R \equiv (\partial_t + \partial_x^3 - \partial_x - \partial_x^{-1}\partial_y^2) u_{ap} + \frac{1}{2}\partial_x(u_{ap}^2).$$

Coming back to (3.6), we obtain that $v(t)$ should solve the problem

$$(\partial_t + \partial_x^3 - \partial_x - \partial_x^{-1}\partial_y^2) v + \partial_x(u_{ap} v) + v\partial_x v + R = 0, \quad v(0) = 0. \tag{3.12}$$

We multiply (3.12) with v and we integrate over $\mathbb{R}\times\mathbb{R}/(2\pi L\mathbb{Z})$. Using integration by parts, we easily get the estimate

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \leq \big(1 + \|\partial_x u_{ap}(t)\|_{L^\infty}\big)\|v(t)\|_{L^2}^2 + \|R(t)\|_{L^2}^2. \tag{3.13}$$

Using (3.10), (3.11) and the Sobolev embedding, we get

$$\|\partial_x u_{ap}(t)\|_{L^\infty} \leq \|Q'\|_{L^\infty} + \sum_{k=1}^{M} C_{k,2}\delta^k e^{k\sigma_0 t}, \quad \|R(t)\|_{L^2}^2 \leq C_M \delta^{2(M+1)} e^{2(M+1)\sigma_0 t}.$$

Coming back to (3.13), we infer that for $t \in [0, T^{\delta}]$,

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \le (D + \kappa\Lambda_M)\|v(t)\|_{L^2}^2 + C_M\delta^{2(M+1)}e^{2(M+1)\sigma_0 t},$$

where $D = 1 + \|Q'\|_{L^\infty}$ and Λ_M, C_M are two positive constants depending of M but independent of κ and t. Integrating the last bound, we get

$$\frac{d}{dt}\Big(e^{-(D+\kappa\Lambda_M)t}\|v(t)\|_{L^2}^2\Big) \le C_M\delta^{2(M+1)}e^{2(M+1)\sigma_0 t - Dt - \kappa\Lambda_M t}, \quad t \in [0, T^{\delta}]. \tag{3.14}$$

We now choose M large enough and κ small enough so that

$$2(M+1)\sigma_0 > D + \kappa\Lambda_M\,. \tag{3.15}$$

This fixes the value of M while κ will be subject to several more smallness restrictions. Thanks to (3.15) we can integrate (3.14) and get the key bound

$$\|v(t)\|_{L^2} \le C_M\kappa^{M+1}, \quad t \in [0, T^{\delta}]. \tag{3.16}$$

We next provide a suitable lower bound for u_{ap}. Let us denote by Π the projector on the nonzero y frequencies. Then for every $a \in \mathbb{R}$, $\Pi(Q(x-a)) = 0$ and by the definition of u_1 there is $c > 0$ such that $\delta\|\Pi u_1(T^{\delta})\|_{L^2} \ge c\kappa$. Using (3.10), we get

$$\big|\big|\Pi u_{ap}(T^{\delta})\big|\big|_{L^2} \ge \frac{c}{2}\kappa, \tag{3.17}$$

provided κ is small enough. Combining (3.16) and (3.17), we get the bound

$$\big|\big|u(T^{\delta}) - Q(x-a)\big|\big|_{L^2} \ge \big|\big|\Pi(u_{ap}(T^{\delta}) + v(T^{\delta}) - Q(x-a))\big|\big|_{L^2} \ge \frac{c}{4}\kappa,$$

provided κ is small enough which implies the instability statement of Theorem 1.3.

We next discuss the stability result of Theorem 1.4. We need to study the stability of $Q_c(x) = cQ(\sqrt{c}x)$ as a solution of

$$\partial_t v - c\partial_x v + v\partial_x v + \partial_x^3 v - \partial_x^{-1}\partial_y^2 v = 0, \quad x \in \mathbb{R},\ y \in \mathbb{T}. \tag{3.18}$$

Consider the energy (Hamiltonian) associated with (3.18)

$$H(v) = \int_{-\infty}^{\infty}\int_0^{2\pi}\Big[(\partial_x v)^2 + (\partial_x^{-1}\partial_y v)^2 + cv^2 - \frac{1}{3}v^3\Big]dxdy.$$

The quantity $H(v)$ is invariant under the flow of (3.18). Using that Q_c is a critical point of H, we can write the expansion

$$H(Q_c + w) = H(Q_c) + B^c(w,w) - \frac{1}{3}\int_{-\infty}^{\infty}\int_0^{2\pi} w^3 dxdy,$$

where

$$B^c(w,w) \equiv 2\int_{-\infty}^{\infty}\int_0^{2\pi}\Big[(\partial_x w)^2 + (\partial_x^{-1}\partial_y w)^2 + cw^2 - Q_c w^2\Big]dxdy.$$

Next, we define the bilinear forms

$$B_k^c(f_1, f_2) \equiv 2\int_{-\infty}^{\infty} \Big[f_1' f_2' + k^2(\partial_x^{-1} f_1)(\partial_x^{-1} f_2) + c f_1 f_2 - Q_c f_1 f_2\Big] dx.$$

We have the following bound for B_0^c.

Lemma 3.2 *There exists $C > 0$ such that for every $c > 0$, every $g \in H^1(\mathbb{R})$ satisfying*

$$\int_{-\infty}^{\infty} g(x) Q_c(x) dx = \int_{-\infty}^{\infty} g(x) Q_c'(x) dx = 0$$

one has

$$B_0^c(g, g) \geq C(\|g'\|_{L^2}^2 + c\|g\|_{L^2}^2).$$

The proof of Lemma 3.2 follows from the KdV stability theory. The next lemma is the key point in the analysis.

Lemma 3.3 *Let $c < 4/\sqrt{3}$. There exists $C > 0$ such that for every $k \in \mathbb{Z}^\star$ and every $f \in H^1(\mathbb{R})$ such that $\partial_x^{-1} f \in L^2(\mathbb{R})$, we have the estimate*

$$B_k^c(f, f) \geq C(\|f\|_{H^1}^2 + k^2\|\partial_x^{-1} f\|_{L^2}^2).$$

The proof of Lemma 3.3 relies on a refined spectral analysis of the operator $\Lambda(k)$, defined above. Note that we do not impose any orthogonality condition in Lemma 3.3. With Lemma 3.2 and Lemma 3.3 in hand we classically complete the stability proof. More precisely, using the implicit function Theorem, we obtain that if the initial data is close to Q_c in Z^1 then there exists a modulation parameter $\gamma(t)$, defined at least for small times, so that

$$v(t, x + \gamma(t), y) = Q_c(x) + w(t, x, y)$$

with

$$\int_{-\infty}^{\infty}\int_0^{2\pi} w(t, x, y) Q_c'(x) dx dy = 0.$$

Recall the conservation law

$$H(v(t)) = H(v(0)) = H(Q_c + w(0)),$$

where $w(0)$ is small in Z^1. On the other hand

$$\begin{aligned} H(v(t)) &= H(Q_c + w(t)) \\ &= H(Q_c) + B^c(w(t), w(t)) - \frac{1}{2}\int_{-\infty}^{\infty}\int_0^{2\pi} w^3(t, x, y) dx dy. \end{aligned}$$

Next, we can write

$$B^c(w(t), w(t)) = B_0^c(\hat{w}(t, \cdot, 0), \hat{w}(t, \cdot, 0)) + \sum_{k \in \mathbb{Z}^\star} B_k^c(\hat{w}(t, \cdot, k), \hat{w}(t, \cdot, k)),$$

where here we use the notation

$$\hat{w}(t,x,k) = (2\pi)^{-1}\int_0^{2\pi} e^{-iky} w(t,x,y)dy,$$

for the partial Fourier transform of w with respect to the periodic variable y. Now, by invoking Lemma 3.2 and Lemma 3.3, we can complete the stability proof. We refer to [27] for the details.

4 On the proof of the results for KP-II

A crucial role in the proof of Theorem 1.6 is played by the Miura transforms which are defined as follows. For $c > 0$, we set

$$M_\pm^c(v) = \pm\partial_x v + \partial_x^{-1}\partial_y v - v^2 + \frac{c}{2},$$

where $v \in Z^1(\mathbb{R}\times\mathbb{T})$. If a sequence $\{v_n\}$ converges to a limit v in Z^1 the sequence $\{M_\pm^c(v_n) - M_\pm^c(v)\}$ converges to 0 in $L^2(\mathbb{R}\times\mathbb{T})$. The key algebraic fact is that if v is a solution of the mKP-II equation

$$\partial_t v + \partial_x^3 v + 3\partial_x^{-1}\partial_y^2 v - 6v^2\partial_x v + 6\partial_x v\partial_x^{-1}\partial_y v = 0$$

then for $c > 0$, $u_\pm$ defined by

$$u_\pm(t,x,y) \equiv M_\pm^c(v)(t,x-3ct,y)$$

are solutions of the KP-II equation

$$\partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\partial_y^2 u = 0. \tag{4.1}$$

We can of course perform the stability analysis in the context of the equation (4.1) and then by a simple scaling this implies the stability statement for (1.7) claimed in Theorem 1.6. Consider the kink Q_c defined by

$$Q_c(x) = \sqrt{\frac{c}{2}}\tanh\left(\sqrt{\frac{c}{2}}x\right).$$

We have that $Q_c(x+ct)$ is a solution of the mKP-II equation and

$$M_+^c(Q_c) = \varphi_c, \quad M_-^c(Q_c) = 0,$$

where

$$\varphi_c(x) \equiv c\cosh^{-2}\left(\sqrt{\frac{c}{2}}x\right), \quad c > 0$$

($\varphi_c(x-2ct)$ is a solution of (4.1)). We next turn to the Cauchy problem for the mKP-II equation for data close to the kink solution. Set

$$Y \equiv \{u \in H^8(\mathbb{R}\times\mathbb{T}) : \partial_x^{-1}\partial_y u \in H^8(\mathbb{R}\times\mathbb{T})\}.$$

We need a global well-posedness result for the mKP-II equation with data

$$v(0,x,y) = Q_c(x) + w_0(x,y), \quad w_0 \in Y. \tag{4.2}$$

It turns out that one can apply arguments similar to the work by Kenig and Martel [9] to get the following result.

Proposition 4.1 *For every $w_0 \in Y$, there exists a unique global in time solution of the mKP-II equation with data* (4.2) *such that*

$$v(t,x,y) = Q_c(x+ct) + w(t,x,y), \quad w \in C(\mathbb{R};Y)\,.$$

We observe that we need the global well-posedness only for regular initial data, the L^2 statement of Theorem 1.6 requires some classical approximation arguments.

Let us now briefly discuss the proof of Proposition 4.1. We need to solve the equation

$$\partial_t w + \partial_x^3 w + 3\partial_x^{-1}\partial_y^2 w - 2\partial_x\big((w+\tilde{Q}_c)^3 - \tilde{Q}_c^3\big) + 6\partial_x w \partial_x^{-1}\partial_y w + 6\tilde{Q}_c'\partial_x^{-1}\partial_y w = 0 \tag{4.3}$$

with data $w(0,x,y) = w_0(x,y)$, $w_0 \in Y$, where $\tilde{Q}_c \equiv Q_c(x+ct)$. It turns out that establishing an L^2 bound for the solutions of (4.3) is a quite delicate task relying on the monotonicity of the kink. Suppose that w is a solution of (4.3) on a time interval $[0,T)$. Define u by $u = M_+^c(\tilde{Q}_c + w)(t, x-3ct, y)$. Then u solves the KP-II equation and by the analysis of [19] we know that for every $s \in [1,6]$ there exists $C_s < \infty$ such that

$$\sup_{t\in[0,T)} \|u(t,\cdot)\|_{H^s(\mathbb{R}_x\times\mathbb{T}_y)} \le C_s.$$

We need to show that there exists $C < \infty$ such that

$$\sup_{t\in[0,T)} \|w(t,\cdot)\|_{L^2} \le C\,.$$

Once the crucial L^2 bound is established, one can also get bounds for higher derivatives. These controls in turn ensure that the local in time analysis of [9] can be suitably iterated in order to get global in time solutions. We have that

$$M_+^c(\tilde{Q}_c + w) = \varphi_c(x+ct) + \partial_x w + \partial_x^{-1}\partial_y w - w^2 - 2\tilde{Q}_c w\,.$$

Thus

$$\sup_{t\in[0,T)} \|\partial_x w + \partial_x^{-1}\partial_y w - w^2 - 2\tilde{Q}_c w\|_{L^2} \le C\,. \tag{4.4}$$

Combining the fact that $(\partial_x w, \partial_x^{-1}\partial_y w) = 0$, $(\partial_x w, w^2) = 0$ and

$$-2(\partial_x w, \tilde{Q}_c w) = \int_{\mathbb{R}_x\times\mathbb{T}_y} Q_c'(x+ct) w^2(x,y)\,dx\,dy > 0,$$

with (4.4), we get

$$\sup_{t\in[0,T)} \|\partial_x w\|_{L^2} + \sup_{t\in[0,T)} \|\partial_x^{-1}\partial_y w - w^2 - 2\tilde{Q}_c w\|_{L^2} \le C\,. \tag{4.5}$$

At that point, we invoke the following Sobolev type inequality

$$\|u\|_{L^6(\mathbb{R}\times\mathbb{T})} \leq C\|\partial_x u\|_{L^2(\mathbb{R}\times\mathbb{T})}^{\frac{1}{3}} \left(\|u\|_{L^2(\mathbb{R}\times\mathbb{T})}^{\frac{2}{3}} + \|\partial_x u\|_{L^2(\mathbb{R}\times\mathbb{T})}^{\frac{1}{3}} \|\partial_x^{-1}\partial_y u\|_{L^2(\mathbb{R}\times\mathbb{T})}^{\frac{1}{3}}\right). \quad (4.6)$$

Observe that if u is y independent, the inequality (4.6) becomes

$$\|u\|_{L^6(\mathbb{R})} \leq C\|\partial_x u\|_{L^2(\mathbb{R})}^{\frac{1}{3}} \|u\|_{L^2(\mathbb{R})}^{\frac{2}{3}}$$

which is a direct consequence of the classical bound $\|f\|_{L^\infty}^2 \leq C\|f'\|_{L^2}\|f\|_{L^2}$. Using (4.6) and the bound for $\|\partial_x w\|_{L^2}$, we get

$$\|\partial_x^{-1}\partial_y w\|_{L^2} \leq C(1 + \|w\|_{L^4}^2 + \|w\|_{L^2}) \leq C(1 + \|w\|_{L^2}^{\frac{1}{2}} \|\partial_x^{-1}\partial_y w\|_{L^2}^{\frac{1}{2}} + \|w\|_{L^2}^2)$$

which in turn implies that for $t \in [0, T)$,

$$\|\partial_x^{-1}\partial_y w\|_{L^2} \leq C(1 + \|w\|_{L^2}^2)\,. \quad (4.7)$$

We next multiply (4.3) by w and integrate over $\mathbb{R}\times\mathbb{T}$ to get the identity

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 = 6\int_{\mathbb{R}\times\mathbb{T}} \tilde{Q}_c\, \partial_x w\, \partial_x^{-1}\partial_y w + 6\int_{\mathbb{R}\times\mathbb{T}} \tilde{Q}_c \tilde{Q}_c' w^2 + 2\int_{\mathbb{R}\times\mathbb{T}} \tilde{Q}_c' w^3.$$

Using (4.6), (4.5) and (4.7), we obtain

$$\Big|\int_{\mathbb{R}\times\mathbb{T}} \tilde{Q}_c' w^3\Big| \leq \|\tilde{Q}_c'\|_{L^2(\mathbb{R}\times\mathbb{T})}\|w\|_{L^6}^3 \leq C(\|w\|_{L^2}^2 + \|\partial_x^{-1}\partial_y w\|_{L^2}) \leq C(1 + \|w\|_{L^2}^2)\,.$$

Using the last estimate and (4.7), we get

$$\frac{d}{dt}\|w\|_{L^2}^2 \leq C(\|w\|_{L^2}^2 + 1)$$

which implies the crucial L^2 control on the solutions of (4.3).

The next key lemma allows to transform the stability problem of the KdV soliton under the KP-II flow to the stability of the kink under the mKP-II flow.

Lemma 4.2 *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|u\|_{L^2} < \delta$, there exists a unique $(k, v) \in \mathbb{R} \times Z^1(\mathbb{R}\times\mathbb{T})$ satisfying*

$$|k - c| < \varepsilon, \quad \|v\|_{Z^1(\mathbb{R}\times\mathbb{T})} < \varepsilon, \quad M_+^k(Q_k + v) = \varphi_c + u.$$

Moreover, the map $u \mapsto (k, v)$ is of class C^1 from $L^2(\mathbb{R}\times\mathbb{T})$ to $\mathbb{R}\times Z^1(\mathbb{R}\times\mathbb{T})$.

An important point of the analysis is the identification of the kernel of the map $\mathcal{L}_c \equiv -\partial_x + \partial_x^{-1}\partial_y - 2Q_c(x)$ which is the linearisation of M_-^c about Q_c. Consider the case $c = 2$ and $Q = Q_2$. Suppose u is in the kernel of $\mathcal{L}_c$. Then it is a solution to a heat equation

$$u_y = (u_x + 2Qu)_x, \quad (4.8)$$

and 2π-periodic in y. A direct computation shows that (4.8) has the y-independent solutions $\{\alpha Q'(x) \mid \alpha \in \mathbb{R}\}$. The key observation is that (4.8) has no other solutions in Z^1. Indeed, let us set

$$V(y) = \int_{\mathbb{R}} \left(\frac{1}{2} u_x^2(x,y) - (Q'(x) - 2Q^2(x)) u^2(x,y) \right) dx.$$

If $u \in Z^1$ is a solution of (4.8) then

$$V'(y) = -\int_{\mathbb{R}} \left(u_y^2(x,y) + Q'(x) \left(u_x(x,y) + 2Q(x) u(x,y) \right)^2 \right) dx.$$

Integrating the last identity over $\mathbb{T}$ yields $u_y = u_x + 2Qu = 0$. Thus u is independent of y and by solving the ODE $u_x + 2Qu = 0$, we obtain that the studied kernel is spanned by Q'.

We now turn to the stability of the kink under the mKP-II flow. We have the following statement.

Proposition 4.3 *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if the initial data*

$$v(0,x,y) = Q_c(x) + w_0(x,y), \quad w_0 \in Y$$

of the mKP-II equation satisfies $\|w_0\|_{Z^1(\mathbb{R}\times\mathbb{T})} < \delta$ then there exists a continuous function $\gamma(t)$ such that for every $t \in \mathbb{R}$, the corresponding solution of the mKP-II equation satisfies

$$||v(t,x,y) - Q_c(x + \gamma(t))||_{Z^1(\mathbb{R}\times\mathbb{T})} < \varepsilon.$$

The proof of Proposition 4.3 can be completed by the arguments described in the end of the previous section once we establish the following bound

$$\|\mathcal{L}_c w\|_{L^2(\mathbb{R}\times\mathbb{T})} \geq C \|w\|_{Z^1}, \quad \forall\, w \in (\varphi_c)^{\perp}, \tag{4.9}$$

which is a quite natural statement in view of the analysis of the kernel of $\mathcal{L}_c$, performed above.

Let us now give the proof of Proposition 4.3, assuming (4.9). Write

$$v(t,x,y) = Q_c(x + \gamma(t)) + w(t,x,y),$$

under the orthogonality condition

$$\left(w(t,x,y), Q_c'(x + \gamma(t)) \right) = \left(w(t,x,y), \varphi_c(x + \gamma(t)) \right) = 0.$$

Set

$$\mathcal{L}_{c,\gamma(t)} \equiv -\partial_x + \partial_x^{-1}\partial_y - 2Q_c(x + \gamma(t)).$$

Recalling that $M_-^c(Q_c) = 0$, we get

$$\begin{aligned} ||M_-^c(Q_c(x+\gamma(t)) + w(t,x,y))||^2_{L^2(\mathbb{R}\times\mathbb{T})} &= \int_{\mathbb{R}\times\mathbb{T}} (\mathcal{L}_{c,\gamma(t)} w - w^2)^2 dxdy \\ &= \int_{\mathbb{R}\times\mathbb{T}} (\mathcal{L}_{c,\gamma(t)} w)^2 dxdy + \int_{\mathbb{R}\times\mathbb{T}} (w^4 - 2w^2 \mathcal{L}_{c,\gamma(t)} w) dxdy. \end{aligned}$$

Thanks to the orthogonality condition, we see that $w(t, x - y(t), y)$ is orthogonal in $L^2(\mathbb{R} \times \mathbb{T})$ to $\varphi_c(x)$. Therefore, using the key property (4.9), we obtain that there exists a positive constant ν, independent of t and w such that

$$\int_{\mathbb{R}\times\mathbb{T}} (\mathcal{L}_{c,y(t)} w)^2 dx dy = \int_{\mathbb{R}\times\mathbb{T}} (\mathcal{L}_c \big(w(t, x - y(t), y) \big)^2 dx dy \geq \nu \|w\|^2_{Z^1(\mathbb{R}\times\mathbb{T})} .$$

Next we invoke (4.6) to get

$$\|u\|_{L^2(\mathbb{R}\times\mathbb{T})} + \|u\|_{L^6(\mathbb{R}\times\mathbb{T})} \leq C\|u\|_{Z^1(\mathbb{R}\times\mathbb{T})}$$

and to arrive at

$$||M^c_-(v)(t, x, y)||^2_{L^2(\mathbb{R}\times\mathbb{T})} \geq \frac{\nu}{2}\|w(t, \cdot)\|^2_{Z^1} - C\|w(t, \cdot)\|^3_{Z^1} .$$

Thanks to the conservation law of the mKP-II flow, we have

$$||M^c_-(v)(t, x, y)||^2_{L^2(\mathbb{R}\times\mathbb{T})} = ||M^c_-(Q_c(x) + w_0(x, y))||^2_{L^2(\mathbb{R}\times\mathbb{T})} .$$

Now expanding the square of the L^2 norm of $M^c_-(Q_c(x) + w_0(x, y))$ and using (4.6), we have

$$||M^c_-(Q_c(x) + w_0(x, y))||^2_{L^2(\mathbb{R}\times\mathbb{T})} \leq C\big(\|w_0\|^2_{Z^1} + \|w_0\|^4_{Z^1}\big).$$

Combining the previous estimates, we get $\|w(t, \cdot)\|_{Z^1} \leq C\|w_0\|_{Z^1}$ provided $\delta \ll 1$. This completes the proof of Proposition 4.3.

The asymptotic stability statement in Theorem 1.6 is based on a use of the fundamental Kato smoothing identity. More precisely, if $u(t) \in C(\mathbb{R}; H^8(\mathbb{R}_x \times \mathbb{T}_y))$ is a solution of the KP-II equation and $\phi(x) \in C^3$ then

$$\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}} u^2\phi = \int_{\mathbb{R}\times\mathbb{T}} \big(-3(\partial_x u)^2 - 3(\partial_x^{-1}\partial_y u)^2 - 4u^3\big)\phi' + \int_{\mathbb{R}\times\mathbb{T}} u^2\phi'''. \qquad (4.10)$$

The identity (4.10) implies that small solutions of the the KP-II equation locally tend to 0 as $t \to \infty$. Finally, thanks to the property $M^c_-(Q_c) = 0$ we can reduce the asymptotic stability close the KdV solitary waves to the asymptotic stability close to zero. We refer to [17] for the details.

5 Related results and open problems

In the context of the transverse stability of the KdV solitary waves as solutions of the KP-II equation a natural question is whether one may consider fully localised perturbations. The global KP-II dynamics for fully localised perturbations is obtained in [19]. In view of the result of [19] one can expect to study the stability phenomenon only locally in space which is the natural counterpart of Theorem 1.6 for fully localised perturbations. In the case of fully localised perturbations new arguments are needed in order to deal with the zero y frequency. We refer to [28] and especially to [16] for more details on this issue.

Let us also mention that the approach of Theorem 2.2 was also used in [15] in order to construct asymptotic mutli-solitons for the water-waves system.

Let us finally mention some open problems related to the results presented here.

1. As already mentioned the transverse stability analysis of the critical speed KdV solitary wave is a delicate problem. The same problem appears in the context of the large family of dispersive models considered in [24]. We believe that in this context the Zakharov-Kuznetsov equation is the most accessible for critical speed transverse stability analysis (see [29] for related results in the context of the nonlinear Schrödinger equation).
2. We believe that it is possible to extend the result of [6] to global well-posedness in Z^1 (see [7, 32]). This would relax the assumption on the perturbation in Theorem 1.4.
3. It would also be very interesting to study the asymptotic stability in the context of Theorem 1.4.
4. We believe that the result of Theorem 1.4 can be extended to a conditional small period stability for the water-waves system (in the spirit of the work by Mielke).
5. We also believe that we can have an unconditional statement in Mielke's analysis for finite but long time scales, depending on the size of the initial perturbation.
6. It would be very interesting to get stability results for the water-waves system in the KP-II regime. For instance, one may try to extend the quite flexible approach of Pego-Weinstein [22] to the case of the water-waves. A first step in this direction is done in [20].

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