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On the long time behavior of the Benjamin-Ono equation

Nikolay Tzvetkov

Cergy-Pontoise University

based on joint works with

Nicola Visciglia and Yu Deng-Nicola Visciglia

(conducted in the period 2009-2014)

Recurrence properties of the KdV equation

Consider the KdV equation, posed on the torus

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0, \quad t \in \mathbb{R}, x \in \mathbb{T},$$

with initial data $u|_{t=0} = u_0 \in H^s(\mathbb{T}; \mathbb{R})$, $s = 0, 1, 2, \dots$

- This problem is globally well-posed (several contributions....)
- KdV models surface waves
- $u : \mathbb{R} \to H^s(\mathbb{T}; \mathbb{R})$ is a continuous curve. How it looks like ?

Theorem 1 (Mc Kean - Trubowitz, Bourgain)
The KdV flow is **almost periodic** *in time : for every ε there exists an almost period l_ε such that for every interval I of size $\geq l_\varepsilon$ there exists $\tau \in I$ such that for every $t \in \mathbb{R}$, $\|u(t+\tau) - u(t)\|_{H^s} < \varepsilon$.*

Corollary 2
The KdV flow is **recurrent** *in time : for* **every** $u_0 \in H^s$ *there is a sequence (t_n) going to infinity such that*

$$\lim_{n \to \infty} \|u(t_n) - u_0\|_{H^s} = 0.$$

Recurrence properties of the Benjamin-Ono equation

Consider the Benjamin-Ono (BO) equation, posed on the torus

$$\partial_t u + H\partial_x^2 u + u\partial_x u = 0, \quad t \in \mathbb{R},\, x \in \mathbb{T},$$

with initial data $u|_{t=0} = u_0 \in H^s(\mathbb{T};\mathbb{R})$, $s = 0, 1, 2, \ldots$

- H denotes the Hilbert transform, i.e. $H(e^{inx}) = -i\text{sgn}(n)e^{inx}$.
- $H^2 = -1$.
- This problem is globally well-posed on L^2 (several contributions on the Cauchy theory, in particular Molinet for L^2)
- BO models internal waves
- Consider the initial data

$$u_0(x,\omega) = \sum_{n\in\mathbb{Z}^\star} \frac{g_n(\omega)}{|n|^{k/2}} e^{inx}, \quad k = 1, 2, 3, 4, 5, 6, 7 \ldots$$

where $g_n(\omega) = h_n(\omega) + il_n(\omega)$, $h_n, l_n \in \mathcal{N}(0,1)$

- $(h_n, l_n)_{n>0}$ are independent and $g_{-n} = \overline{g_n}$.
- $u_0 \in H^s$, a.s. for $s < \frac{k-1}{2}$ but $u_0 \notin H^{\frac{k-1}{2}}$ a.s.

Theorem 3 (NT and Nicola Visciglia (2009-2013))

For almost every ω and for every $k \geq 4$ the solution of the Benjamin-Ono equation with data given by

$$u_0(x,\omega) = \sum_{n \in \mathbb{Z}^\star} \frac{g_n(\omega)}{|n|^{k/2}} e^{inx}$$

is recurrent in time with convergence in H^s, $s < \frac{k-1}{2}$.

A deterministic corollary

Corollary 4

Fix an integer $s \geq 0$. Then there exists a dense set F_s of $H^s(\mathbb{T};\mathbb{R})$ such that for every $u_0 \in F_s$ the solution of the Benjamin-Ono equation with data u_0 is recurrent.

• Question 1 : Can we take $F_s = H^s(\mathbb{T};\mathbb{R})$?

• Question 2 : Is the Benjamin-Ono flow posed on $\mathbb{T}$ almost periodic in time, at least for small data (Coifman and Wickerhauser on the line $\mathbb{R}$)?

• We get a recurrence property for data which is **not small** and which is not of **low regularity**.

• A similar work for KdV was done by Zhidkov.

Invariant measures for the Benjamin-Ono equation

• We prove our results by constructing invariant measures.

• There is an infinite sequence of conservation laws satisfied by the solutions of the Benjamin-Ono equation (our reference is a book by Matsuno) : if u is a smooth solution of BO then :

$$\frac{d}{dt}E_{k/2}(u(t)) = 0, \quad k = 0, 1, 2, 3, \ldots$$

where

$$E_{k/2}(u) = \||\partial_x|^{k/2}u\|_{L^2}^2 + R_{k/2}(u),$$

• $R_{k/2}(u)$ is a sum of terms homogenous in u of order ≥ 3 (but containing less derivatives).

Here is the list of the first conservation laws :

$$
\begin{aligned}
E_0(u) &= \|u\|_{L^2}^2; \\
E_{1/2}(u) &= \||\partial_x|^{1/2}u\|_{L^2}^2 + \frac{1}{3}\int u^3 dx; \\
E_1(u) &= \|\partial_x u\|_{L^2}^2 + \frac{3}{4}\int u^2 H(u_x) dx + \frac{1}{8}\int u^4 dx; \\
E_{3/2}(u) &= \||\partial_x|^{3/2}u\|_{L^2}^2 - \int [\frac{3}{2}u(u_x)^2 + \frac{1}{2}u(Hu_x)^2] dx \\
&\quad - \int [\frac{1}{3}u^3 H(u_x) + \frac{1}{4}u^2 H(uu_x)] dx - \frac{1}{20}\int u^5 dx; \\
E_2(u) &= \|\partial_x^2 u\|_{L^2}^2 - \frac{5}{4}\int [(u_x)^2 Hu_x + 2uu_{xx}Hu_x] dx \\
&\quad + \frac{5}{16}\int [5u^2(u_x)^2 + u^2 H(u_x)^2 + 2uH(\partial_x u)H(uu_x)] dx \\
&\quad + \int [\frac{5}{32}u^4 H(u_x) + \frac{5}{24}u^3 H(uu_x)] dx + \frac{1}{48}\int u^6 dx
\end{aligned}
$$

Construction of the measures

• The basic idea (Lebowitz-Rose-Speer) is to renormalize the formal measure $\exp(-E_{k/2}(u))du$ by seeing $\exp(-\||\partial_x|^{k/2}u\|^2_{L^2})du$ as a gaussian measure on a suitable Hilbert space and to see $\exp(-R_{k/2}(u))$ as an integrable density.

• Therefore, we define $\mu_{k/2}$ as a measure on H^s, $s < \frac{k}{2} - \frac{1}{2}$, by the map

$$\omega \longmapsto \varphi_{k/2}(x, \omega),$$

where

$$\varphi_{k/2}(x, \omega) = \sum_{n \neq 0} \frac{g_n(\omega)}{|n|^{k/2}} e^{inx},$$

with

$$g_{-n} = \overline{g_n}, \quad g_n = c(h_n + il_n), \quad h_n, l_n \in \mathcal{N}(0, 1)$$

and $(h_n, l_n)_{n>0}$ are independent.

• Any set of full $\mu_{k/2}$ measure is dense in $H^s|\{\text{const}\}$.

• But $\mu_{k/2}(H^{\frac{k}{2}-\frac{1}{2}}) = 0$.

• Denote by π_N the Dirichlet projector ($\pi_N(\sum_n c_n e^{inx}) = \sum_{|n|\leq N} c_n e^{inx}$).

• Let χ_R be a cut-off function defined as $\chi_R(x) = \chi(x/R)$ with $\chi : \mathbb{R} \to \mathbb{R}$ a smooth, compactly supported function such that $\chi(x) = 1$ for every $|x| < 1$.

• For $N \geq 1$, $k \geq 1$ and $R > 0$ we introduce the function

$$F_{k/2,N,R}(u) = \Big(\prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N u))\Big)\chi_R(E_{(k-1)/2}(\pi_N u) - \alpha_N)e^{-R_{k/2}(\pi_N u)}$$

where $\alpha_N = \sum_{n=1}^{N} \frac{c}{n} \approx \log(N)$ for a suitable constant c.

Theorem 5 (NT and Nicola Visciglia 2010)

For every $k \geq 1$, for every $R > 0$ there exists a $\mu_{k/2}$ measurable function $F_{k/2,R}(u)$ such that $F_{k/2,N,R}(u)$ converges to $F_{k/2,R}(u)$ in $L^q(d\mu_{k/2})$ for every $1 \leq q < \infty$. In particular $F_{k/2,R}(u) \in L^q(d\mu_{k/2})$. Moreover, if we set $d\rho_{k/2,R} \equiv F_{k/2,R}(u)d\mu_{k/2}$ then we have

$$\bigcup_{R>0} \text{supp}(\rho_{k/2,R}) = \text{supp}(\mu_{k/2})$$

• A main tool in the proof of the above result is the classical heat flow estimate

$$\|e^{t(\Delta_{\mathbb{R}^d}-x\cdot\nabla_{\mathbb{R}^d})}(f)\|_{L^p(\mathbb{R}^d;(2\pi)^{-d/2}e^{-|x|^2/2}dx)} \leq \|f\|_{L^2(\mathbb{R}^d;(2\pi)^{-d/2}e^{-|x|^2/2}dx)},$$

provided

$$p \geq 2, \quad t \geq \frac{1}{2}\log(p-1)\,.$$

• Are the measures $\rho_{k/2,R}$ indeed invariant by the BO flow ?

Theorem 6 (NT and Nicola Visciglia (2009-2013))
Denote by $\Phi_t : H^s \to H^s$, $s \geq 0$, the flow of the Benjamin-Ono equation. Then $\rho_{k/2,R}$, $k \geq 4$, are invariant under Φ_t :

$$\rho_{k/2,R}(A) = \rho_{k/2,R}(\Phi_t(A)), \quad \forall\, t \in \mathbb{R}$$

and every measurable set A.

Theorem 7 (Yu Deng 2013)
The above result holds true for $k = 1$ (by suitably defining Φ_t).

Theorem 8 (Yu Deng, NT and Nicola Visciglia 2014)
The above result holds true for $k = 2, 3$.

Comments

• The main difficulty in proving such a result for $k \geq 2$ is that the conservation laws are no longer conserved under truncated versions of the equation. The new argument with respect to previous works on invariant measures is that we reduce the matters to a property at $t = 0$ (as in the proof of Liouville's theorem). In particular, we do not need to evaluate the energy growth of individual solutions as in the work by Zhidkov, Oh, Nahmod-Oh-Rey-Bellet-Staffilani ...

• We also use an algebraic "miracle" related to the structure of the conservation laws.

• The impressive work by Yu Deng treates the case $k = 1$. In this case one does not need to resolve the above difficulty since the Hamiltonian is conserved under truncated versions of the equation. But there is a major regularity problem to be solved, namely an improvement on the Molinet result is obtained (one covers the support of $\mu_{1/2}$ which misses L^2)

On the invariance proof for $k \geq 4$

• For $N \geq 1$, consider the approximated problem

$$\partial_t u + H\partial_x^2 u + \pi_N\Big(\pi_N u\, \partial_x \pi_N u\Big) = 0,$$

with a corresponding flow on H^s denoted by Φ_t^N.

• We have the following approximation property between Φ_t and Φ_t^N (a traditional dispersive PDE analysis becoming harder at low regularities) :

$$\exists s < \sigma < (k-1)/2 \text{ s.t. } \forall\, r > 0,\ \exists\, \bar{t} = \bar{t}(r) > 0 \text{ s.t. } \forall\, \varepsilon > 0,$$
$$\Phi_t^N(A) \subset \Phi_t(A) + B^s(\varepsilon),\ \forall\, N > N_0(\varepsilon), \forall\, t \in (-\bar{t}, \bar{t}), \forall A \subset B^\sigma(r),$$

where $B^\sigma(r)$ denotes the ball of radius r and centered at the origin of the Sobolev space H^σ.

On the invariance proof (sequel)

• Set $E_N = \text{span}(\cos(nx), \sin(nx))_{1\le n\le N}$. Then

$$d\mu_{k/2}(u) = \gamma_N e^{-\|\pi_N u\|^2_{H^{k/2}}} du_1 \dots du_N \; d\mu_N^{\perp},$$

where $u \in E_N$ and $d\mu_N^{\perp}$ is a gaussian measure on $E_N^{\perp}$.

• Using the approximation properties between Φ_t and Φ_t^N, we need to show that there exists $s < \frac{k-1}{2}$ such that for every $t_0 \in \mathbb{R}$,

$$\lim_{N\to\infty} \sup_{\substack{t\in[0,t_0] \\ A\in\mathcal{B}(H^s)}} \left| \frac{d}{dt} \int_{\Phi_t^N(A)} F_{k/2,N,R}(u) d\mu_{k/2}(u) \right| = 0$$

On the invariance proof (sequel)

• By the invariance of the Lebesque measure under the flow of divergence free vector fields and the invariance of complex gaussians under rotations, we get

$$\int_{\Phi_t^N(A)} F_{k/2,N,R}(u)d\mu_{k/2}(u) = \gamma_N \int_A \prod_{j=0}^{k-2} \chi_R(E_{j/2}(\pi_N \Phi_t^N(u)))\times$$
$$\chi_R(E_{(k-1)/2}(\pi_N \Phi_t^N(u)) - \alpha_N) e^{-E_{k/2}(\pi_N(\Phi_t^N(u))} du_1 ... du_N \times d\mu_N^{\perp}$$

• We can reduce the problem to $t = 0$ (this is an important point).
• Therefore, by Cauchy-Schwarz, we need to evaluate the L^2 norms with respect to $\mu_{k/2}$ of the quantities

$$L_N^j(u) = \frac{d}{dt}\Big(E_{j/2}\Big(\pi_N \Phi_t^N(u)\Big)\Big)\Big|_{t=0}, \quad 0 \le j \le k/2.$$

On the invariance proof (sequel)

• Consequently, the key property is :

$$\lim_{N\to\infty} \|L_N^j(u)\|_{L^q(d\mu_{k/2}(u))} = 0, \quad q < \infty, \tag{1}$$

$$L_N^j(u) = \frac{d}{dt}\Big(E_{j/2}\big(\pi_N \Phi_t^N(u)\big)\Big)\Big|_{t=0}, \quad 0 \le j \le k/2.$$

• The quantity $L_N^j(u)$ can be expressed quite explicitly in terms of the random series

$$\sum_{n\neq 0} \frac{g_n(\omega)}{|n|^{k/2}}\, e^{inx}\,.$$

• The proof of (1) uses the fine algebraic structure of the conservation laws of the Benjamin-Ono equation and in particular the precise positions of the Hilbert transforms in the densities defining the conservation laws.

On the proof of the key property

• One is reduced to prove that the $L^2(\omega)$ norm of expressions of the following type go to zero as long as $N \to \infty$:

$$\sum_{\mathcal{C}_N} c_{j_1,...,j_n} g_{j_1}(\omega) \times ... \times g_{j_n}(\omega)$$

where $c_{j_1,...,j_n}$ are suitable numbers.

• In the case $k \geq 6$ and k even via the triangle inequality we are reduced to the analysis of series of the type

$$\sum_{\mathcal{C}_N} |c_{j_1,...,j_n}|$$

• If $k = 2, 4$ and $k \geq 3$ odd then the triangle inequality is useless and one exploit the $L^2(\omega)$ orthogonality $g_{j_1}(\omega) \times ... \times g_{j_n}(\omega)$ and we reduce the analysis to expressions of the type

$$\sum_{\mathcal{C}'_N} |c_{j_1,...,j_n}|^2$$

where $\mathcal{C}'_N$ is a large subset of $\mathcal{C}_N$. The analysis on the resonant set $\mathcal{C}_N \backslash \mathcal{C}'_N$ is then done again via the triangle inequality.

The cases $k = 2, 3$

We denote by $\Phi_t^{\epsilon,N}(u_0)$ the solution to

$$\partial_t u + H\partial_x^2 u + S_N^\epsilon(S_N^\epsilon u \cdot S_N^\epsilon u_x) = 0, u(0) = u_0$$

and S_N^ϵ are smoothed Dirichlet projectors, defined by

$$S_N^\epsilon(\sum_{j\in\mathbb{Z}} a_j e^{ijx}) = \sum_{j\in\mathbb{Z}} a_j \psi_\epsilon(\frac{j}{N}) e^{ijx},$$

where for $\epsilon \in (0,1)$, ψ_ϵ is a smooth function $\psi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that

$$\psi_\epsilon(x) = 1 \text{ for } x \in [0, (1-\epsilon)], \psi_\epsilon(x) = 0 \text{ for } x > 1,$$

$$\|\psi_\epsilon\|_{L^\infty} = 1 \text{ and } \psi_\epsilon(x) = \psi_\epsilon(|x|).$$

On the construction of the measure $d\rho_{1,R}$ via S_N^ϵ

Take $k = 2$. Define the smoothed approximating measures

$$F^\epsilon_{N,R} = \chi_R(\|\pi_N u\|_{L^2}) \times \chi_R(\|\pi_N u\|^2_{\dot H^{1/2}} - \alpha_N + 1/3 \int (S^\epsilon_N u)^3 dx)$$
$$\times \exp(-\frac{3}{4}\int (S^\epsilon_N u)^2 H\partial_x S^\epsilon_N u - \frac{1}{8}\int (S^\epsilon_N u)^4)$$

Proposition 9

The following occurs for every $\epsilon > 0, \sigma > 0$:

$$\lim_{N\to\infty} \sup_{A\in\mathcal{B}(H^{1/2-\sigma})} |\int_A F^\epsilon_{N,R} d\mu_1 - \int_A d\rho_{1,R}| = 0.$$

• Roughly speaking this means that the limit measure $d\rho_{1,R}$ does not depend on the fixed value of $\epsilon > 0$. Indeed, we get in the limit as $N \to \infty$ the same measure that we get when we approximating measures involve the sharp projectors π_N.

Proposition 10

Let $0 < \epsilon < 1$, $\sigma > \sigma' > 0$ *and* $M > 0$ *be fixed, then for some* $T = T(\epsilon, \sigma, \sigma', M) > 0$, $C = C(\epsilon, \sigma, \sigma', M) > 0$ *we get:*

$$\sup_{\phi \in B_M(H^{1/2-\sigma'})} \sup_{|t| \leq T} \|\Phi_t^{\epsilon,N}\phi - \Phi_t\phi\|_{H^{1/2-\sigma}} \leq CN^{-\theta},$$

where $\theta = \theta(\sigma, \sigma') > 0$.

Proposition 11

$$\lim_{\epsilon \to 0} \Big(\limsup_{N \to \infty} \| \frac{d}{dt} E_1(\pi_N \Phi_t^{\epsilon,N})_{t=0} \|_{L^2(d\mu_1)} \Big) = 0 \Rightarrow$$

$$\forall\, \delta > 0,\ \exists\, \epsilon = \epsilon(\delta) > 0, N = N(\delta) > 0 \textit{ s.t.}$$

$$| \int_A F_{N,R}^{\epsilon} d\mu_1 - \int_{\Phi_t^{\epsilon,N}(A)} F_{N,R}^{\epsilon} d\mu_1 | \leq \delta t.$$

• We need $\epsilon \to 0, N \to \infty$. This is not necessary for $E_{1/2}, E_{k/2}, k \geq 4$.
• As long as $\epsilon \to 0$ then the time of approximation of the smoothed truncated flows to the true solution of BO becomes smaller and smaller.

Proof of the invariance of $d\rho_{1,R}$

• Fix $\bar{t} \in \mathbb{R}$. We prove $\int_A d\rho_{1,R} \leq \int_{\Phi_{\bar{t}}(A)} d\rho_{1,R}, \forall A \subset H^{1/2-\sigma}$, compact.
• Thanks to the second proposition, for every $k > 0$ we get $N_k \in \mathbb{N}$ and $\epsilon_k > 0$ such that:

$$|\int_A F_{N,R}^{\epsilon_k} d\mu_1 - \int_{\Phi_t^{\epsilon_k,N}(A)} F_{N,R}^{\epsilon_k} d\mu_1| \leq t/k, \qquad \forall N > N_k, \quad \forall t.$$

• Thanks to the flows approximation property $\exists t_1 = t_1(k) > 0$ s.t.

$$\int_{\Phi_t^{\epsilon_k,N}(A)} F_{N,R}^{\epsilon_k} d\mu_1 \leq \int_{\Phi_t(A)+B_{CN^{-\theta}}(H^{1/2-\sigma'})} F_{N,R}^{\epsilon_k} d\mu_1, \ \forall t \in [0, t_1]$$

and we get

$$\int_A F_{N,R}^{\epsilon_k} d\mu_1 \leq \int_{\Phi_t(A)+B_{CN^{-\theta}}(H^{1/2-\sigma'})} F_{N,R}^{\epsilon_k} d\mu_1 + t_1/k, \ \forall t \in [0, t_1].$$

In the limit $N \to \infty$, we get :

$$\int_A d\rho_{1,R} \leq \int_{\Phi_t(A)} d\rho_{1,R} + t_1/k, \ \forall t \in [0, t_1].$$

Iterate the bound $[\bar{t}/t_1] + 1$ times and take the limit as $k \to \infty$ closes the argument.

Final remarks

We hope that this approach can be useful in other contexts ...